About dual two-dimensional oscillator and Coulomb-like theories on a plane

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Abstract

We present a mathematically rigorous quantum-mechanical treatment of a two-dimensional nonrelativistic quantum dual theories (with oscillator and Coulomb like potentials) on a plane and compare their spectra and the sets of eigenfunctions. All self-adjoint Schrodinger operators for these theories are constructed and rigorous solutions of the corresponding spectral problems are presented. The first part of the problem is solved by using a method of specifying s.a. extensions by (asymptotic) s.a. boundary conditions. Solving spectral problems, we follow the Krein's method of guiding functionals. We show, that there is one to one correspondence between the spectral points of dual theories in the planes energy-coupling constants not only for discrete, but also for continuous spectra.

1 Introduction

It is well known [1], that if one introduces in a radial part of the D dimensional oscillator ($D > 2$)

$$\frac{d^2R}{du^2} + \frac{D-1}{u}\frac{dR}{du} - \frac{L\,(L+D-2)}{u^2}R + \frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega^2u^2}{2}\right)R = 0 \tag{1.1}$$

(here R is the radial part of the wave function for the D dimensional oscillator ($D > 2$) and $L = 0, 1, 2, ...$ are the eigenvalues of the global angular momentum) $r = u^2$ then equation (1.1) transforms into

$$\frac{d^2R}{dr^2} + \frac{d-1}{r}\frac{dR}{dr} - \frac{l\,(l+d-2)}{r^2}R + \frac{2\mu}{\hbar^2}\left(\mathcal{E} + \frac{\alpha}{r}\right)R = 0 \tag{1.2}$$

where $d = D/2+1 \quad l = L/2 \quad \mathcal{E} = -\frac{\mu\omega^2}{8} \quad \alpha = E/4$, which formally is identical to the radial equation for d-dimensional hydrogen atom.

Equations (1.1) and (1.2) are dual to each other and the duality transformation is $r = u^2$. For discreet spectrum of these equations (and wave fuctions regular at the origin) it was proved, that to each state of equation (1.1) corresponds a state in (1.2), and visa versa [2, 3]. However the correspondence of the states in general (for discrete, as well as continuous spectra and for all values of the parameters of the theory) the problems was not considered.

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In [4] we constructed all self-adjoint Schrodinger operators for nonrelativistic one-dimensional quantum dual theories and represented rigorous solutions of the corresponding spectral problems. We have shown that there is one to one correspondence between the spectra of dual theories for discreet , as well as continuous spectra.

In this paper will solve the quantum problem of two dimensional quantum dual theories (with oscillator and Coulomb like potentials) on a plane and compare their spectra and the sets of eigenfunctions. As it was in one dimensional case, we again have a correspondence of the states for all values of the parameters E_O, λ, E_C, and g, except when the angular momentum $m = 1$, when the duality is one-to-one only in the case of parameter of s.a. extension $\zeta = \pi/2$ (see below in coulomb case). We have similar situation for dual models on two dimensional pseudoshpere. The interest to these models was stimulated also by the fact that among the theorists dealing with similar problems exists a notion, that the "Hamiltonian isn't self adjoit at high energies" [5]. In section 2 we will consider the quantum problem for the oscillator, will find solutions of the equation for all values of the variable and parameters. In Section 3 we will consider the quantum problem for Coulomb-like system. The results will be compared in section 4, where we will show the one-to one correspondense of the spectra and proper functions of the Hamiltonians of both problems.

2 Quantum two-dimentional oscillator-like interaction on a plane

2.1 Reduction to radial problem

In the case under consideration, the initial differential operation is $\check{H}_O \equiv \check{H}$,

$$\check{H} = -\Delta + \lambda\,\mathbf{u}^2,\ \mathbf{u} = (u_x, u_y),\ \Delta = \partial^2_{u_x} + \partial^2_{u_y} = \partial^2_u + \frac{1}{u}\partial_u + \frac{1}{u^2}\partial^2_{\varphi_u},$$
$$u = \sqrt{u_x^2 + u_y^2},$$

the space of particle quantum states is the Hilbert space $\mathfrak{H}_O \equiv \mathfrak{H} = L^2(\mathbb{R}^2)$ of s.-integrable functions $\psi(\mathbf{u})$, , with the scalar product

$$(\psi_1, \psi_2) = \int \overline{\psi_1(\mathbf{u})}\psi_2(\mathbf{u})d\rho,\ d\mathbf{u} = du_x du_y = udud\varphi_u,\ u \in \mathbb{R}_+,\ \varphi_u \in [0, 2\pi].$$

A quantum Hamiltonian should be defined as a s.a. operator in this Hilbert space.

The construction is essentially based on the requirement of rotation symmetry which certainly holds in a classical description of the system. This requirement is formulated as the requirement of the invariance of a s.a. Hamiltonian under rotations around the origin. As in classical mechanics, the rotation symmetry allows separating the polar coordinates ρ and φ and reducing the two-dimensional problem to a one-dimensional radial problem.

The group of rotations $SO(2)$ in $\mathbb{R}^2$ naturally acts in the Hilbert space $\mathfrak{H}$ by unitary operators: if $S \in SO(2)$, then the corresponding operator $\hat{U}_S$ is defined by the relation $(\hat{U}_S\psi)(\mathbf{u}) = \psi(S^{-1}\mathbf{u})$, $\psi \in \mathfrak{H}$.

The Hilbert space $\mathfrak{H}$ is a direct orthogonal sum of subspaces $\mathfrak{H}_m$, that are the eigenspaces of the representation $\hat{U}_S$,

$$\mathfrak{H} = \sum_{m\in\mathbb{Z}}^{\oplus} \mathfrak{H}_m,\ \hat{U}_S\mathfrak{H}_m = e^{-im\theta}\mathfrak{H}_m,\ \mathfrak{H}_m = \hat{P}_m\mathfrak{H}$$

where θ is the rotation angle corresponding to S and .$\hat{P}_m$ is an orthohonal projector on subspace $\mathfrak{H}_m$. $\mathfrak{H}_m$ consists of eigenfunctions $\psi_m(\mathbf{u})$ for angular momentum operator $\hat{L}_z = -i\hbar\partial/\partial\varphi_u$, $\psi_m(\mathbf{u}) = \frac{1}{\sqrt{u}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}f_m(u)$, where $f_m(u) \in \mathfrak{h}_{Om} \equiv \mathfrak{h}_m = L^2(\mathbb{R}_+)$, $L^2(\mathbb{R}_+)$ is the Hilbert space of s.-integrable functions on the semi-axis $\mathbb{R}_+$ with scalar product

$$(f,g) = \int_{\mathbb{R}_+} \overline{f(u)}g(u)\,du.$$

We define an initial symmetric operator $\hat{H}_O \equiv \hat{H}$ associated with $\check{H}$ as follows:

$$:\left\{\begin{array}{l} D_H = \{\psi(\mathbf{u}) : \ \psi \in \mathcal{D}(\mathbb{R}^2\backslash\{0\})\} \\ \hat{H}\psi = \check{H}\psi, \ \forall\psi \in D_H \end{array}\right. ,$$

where $\mathcal{D}(\mathbb{R}^2\backslash\{0\})$ is the space of smooth and compactly supported functions vanishing in a neighborhood of the point $\mathbf{u}$ =0. The domain D_H is dense in $\mathfrak{H}$ and the symmetricity of $\hat{H}$ is obvious. It is also obvious that the operator $\hat{H}$ commutes[1] with the unitary operators $\hat{U}_S$,.

$$\hat{H} = \sum_{m\in\mathbb{Z}}^{\oplus} \hat{H}_m, \ \hat{H}_m = \hat{P}_m\hat{H}.$$

Operators $\hat{f}$ which commute with the operators $\hat{U}_S$, we will call rotationally-invariant. Such operators can be represented in the form

$$\hat{f} = \sum_{m\in\mathbb{Z}}^{\oplus} \hat{f}_m, \ \hat{f}_m = \hat{P}_m\hat{f},$$

and $\hat{f}_m$ are s.a. operators in sufspaces $\mathfrak{H}_m$.

In the polar coordinates u, φ_u the operation $\check{H}$ becomes

$$\check{H} = -\partial_u^2 - u^{-1}\partial_u + -u^{-2}\partial_{\varphi_u}^2 + \lambda u^2,$$

Represent $\psi(\mathbf{u}) \in \mathfrak{H}$ in the form

$$\psi(\mathbf{u}) = \sum_{m\in\mathbb{Z}} \psi_m(\mathbf{u}), \ \psi_m(\mathbf{u}) = \frac{1}{\sqrt{u}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}f_m(u), \ f_m(u) = \sqrt{u}\int_0^{2\pi} d\varphi_u \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-im\varphi_u}\psi(\mathbf{u}).$$

Then we have

$$\hat{H}\psi_m(\mathbf{u}) = \hat{H}_m\psi_m(\mathbf{u}) = \frac{1}{\sqrt{u}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\hat{h}_{Om}f_m(u),$$

where $\hat{h}_{Om} \equiv \hat{h}_m$ is a symmetric operator defined in the Hilbert space $\mathfrak{h}_m$:

$$\hat{h}_m : \left\{\begin{array}{c} D_{h_m} = \{f(u) : f(u) \in \mathcal{D}(\mathbb{R}_+)\} \\ \hat{h}_m f = \check{h}_m f, \ \forall f \in D_{h_m}, \ \check{h}_m = -\partial_u^2 + u^{-2}(m^2 - 1/4) + \lambda u^2 \end{array}\right. .$$

Let $\hat{h}_{\mathfrak{e}m}$ is a s.a.operator associated with the differential operation $\check{h}_m$.in the Hilbert space $\mathfrak{h}_m$ Then the operator $\hat{H}_{\mathfrak{e}m}$,

$$\hat{H}_{\mathfrak{e}m}\psi_m(\mathbf{u}) = \frac{1}{\sqrt{u}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\hat{h}_{\mathfrak{e}m}f_m(u), \ f_m(u) \in D_{h_{\mathfrak{e}m}}, \tag{2.1}$$

[1]We remind the reader of the notion of commutativity in this case (where one of the operators, U_S, is bounded and defined everywhere): we say that the operators $\hat{H}$ and U_S commute if $U_S\hat{H} \subseteq \hat{H}U_S$, i.e., if $\psi \in D_H$, then also $U_S\psi \in D_H$ and $U_S\hat{H}\psi = \hat{H}U_S\psi$

is a s.a. operator operator associated with $\check{H}_m$.in the Hilbert space $\mathfrak{H}_m$ and operator $\hat{H}_\mathfrak{e}$,

$$\hat{H}_\mathfrak{e} = \sum_{m\in\mathbb{Z}}^{\oplus} \hat{H}_{\mathfrak{e}m}, \tag{2.2}$$

is a s.a. operator in the Hilbert space $\mathfrak{H}$.

Conversely, let $\hat{H}_\mathfrak{e}$ be a rotationally invariant s.a. extension of $\hat{H}$. Then it has the form (2.2), where $\hat{H}_{\mathfrak{e}m}$ are s.a. operators in $\mathfrak{H}_m$. The operator $\hat{H}_{\mathfrak{e}m}$ acts in subspace $\mathfrak{H}_m$ by the rule (2.1) with some operator $\hat{h}_{\mathfrak{e}m}$ which is obviously a s.a. operator associated with the symmetric operator $\hat{h}_m$ in the Hilbert space $\mathfrak{h}_m$.

Thus, the problem of constructing a rotationally-invariant s.a. Hamiltonian $\hat{H}_\mathfrak{e}$ is thus reduced to constructing s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}\mathfrak{m}}$.

2.2 $|m| \geq 1,\ \lambda > 0$

2.3 Useful solutions

We need solutions of an equation

$$(\check{h}_m - W)\psi(u) = 0,\ \check{h}_m = -\partial_u^2 + u^{-2}(m^2 - 1/4) + \lambda u^2, \tag{2.3}$$

where $\hbar^2 W/2m$ is complex energy, $\hbar^2\lambda/2m$ is a coupling constant,

$$W = |W|e^{i\varphi_W},\ 0 \leq \varphi_W \leq \pi,\ \operatorname{Im} W \geq 0,$$

and for λ, we will use the representation $\lambda = \varkappa^4$.

It is convenient for our aims first to consider solutions more general equation

$$[-\partial_u^2 + u^{-2}((m+\delta)^2 - 1/4) + \lambda u^2 - W]\psi(u) = 0,\ |\delta| < 1. \tag{2.4}$$

Introduce a new variable $\rho = (\varkappa u)^2$, $\partial_u = 2\varkappa\sqrt{\rho}\partial_\rho$, $\partial_u^2 = 4\varkappa^2[\rho\partial_\rho^2 + (1/2)\partial_\rho]$, and new function $\phi(\rho)$, $\psi(u) = \rho^{1/4+|m+\delta|/2}e^{-\rho/2}\phi(\rho)$. Then we obtain

$$\begin{aligned} &\rho\partial_\rho^2\phi(\rho) + (\beta_\delta - \rho)\partial_\rho\phi(\rho) - \alpha_\delta\phi(\rho) = 0,\ \beta_\delta = 1 + |m+\delta|, \\ &\alpha_\delta = 1/2 + |m+\delta|/2 - w,\ w = w_O = W/4\varkappa^2. \end{aligned} \tag{2.5}$$

Eq. (2.5) is the equation for confluent hypergeometric functions, in the terms of which we can express solutions of eq. (2.4). We will use the following solutions

$$O_{1,m,\delta}(u;W) = (\kappa_0 u)^{1/2+|m+\delta|}e^{-\rho/2}\Phi(\alpha_\delta,\beta_\delta;\rho), \tag{2.6}$$

$$O_{2,m,\delta}(u;W) = \frac{(\kappa_0 u)^{1/2-|m+\delta|}}{\Gamma(\beta_{-,\delta})}e^{-\rho/2}\Phi(\alpha_{-,\delta},\beta_{-,\delta};\rho), \tag{2.7}$$

$$\begin{aligned} &O_{3,m,\delta}(u;W) = (\kappa_0 u)^{1/2+|m+\delta|}e^{-\rho/2}\Psi(\alpha_\delta,\beta_\delta;\rho) = \\ &= \frac{\Gamma(-|m+\delta|)}{\Gamma(\alpha_{-,\delta})}O_{1,m,\delta}(u;W) + \frac{(\kappa_0/\varkappa)^{2|m+\delta|}\Gamma(|m+\delta|)\Gamma(\beta_{-,\delta})}{\Gamma(\alpha_\delta)}O_{2,m,\delta}(u;W), \\ &\alpha_{-,\delta} = 1/2 - |m+\delta|/2 - w,\ \beta_{-,\delta} = 1 - |m+\delta|. \end{aligned}$$

Note that $O_{1,m,\delta}(u;W)$ and $O_{2,m,\delta}(u;W)$ are real-entire in W solutions of eq. (2.4).

Represent $O_{3,m,\delta}$ in the form

$$O_{3,m,\delta}(u;W) = B_{m,\delta}(W)O_{1,m,\delta}(u;W) + \frac{(\kappa_0/\varkappa)^{2|m+\delta|}\Gamma(|m+\delta|)}{\Gamma(\alpha_\delta)}O_{4,m,\delta}(u;W), \tag{2.8}$$

$$O_{4,m,\delta}(u;W) = \Gamma(\beta_{-,\delta})\left[O_{2,m,\delta}(u;W) - A_{m,\delta}(W)O_{1,m,\delta}(u;W)\right], \tag{2.9}$$

$$A_{m,\delta}(W) = \frac{(\kappa_0/\varkappa)^{-2|m|}\Gamma(\alpha)}{\Gamma(\beta_\delta)\Gamma(\alpha_-)},\ \frac{\Gamma(\alpha)}{\Gamma(\alpha_-)} = (-1)^{|m|}(1-\alpha)_{|m|},\ (1+x)_{|m|} = (1+x)\cdots(|m|+x), \tag{2.10}$$

$$B_{m,\delta}(W) = \frac{\Gamma(-|m+\delta|)}{\Gamma(\alpha_{-,\delta})} + \frac{(\kappa_0/\varkappa)^{2|m+\delta|}\Gamma(|m+\delta|)\Gamma(\beta_{-,\delta})}{\Gamma(\alpha_\delta)}A_{m,\delta}(W) =$$
$$= \frac{\Gamma(-|m+\delta|)}{\Gamma(\alpha_-)}\left[\frac{\Gamma(\alpha_-)}{\Gamma(\alpha_{-,\delta})} - \frac{(\kappa_0/\varkappa)^{2\delta_m}\Gamma(\alpha)}{\Gamma(\alpha_\delta)}\right], \tag{2.11}$$
$$\alpha = \alpha_0 = 1/2 + |m|/2 - w,\ \alpha_- = \alpha_{-,0} = 1/2 - |m|/2 - w,\ \beta = \beta_0 = 1 + |m|,$$

$O_{4,m,\delta}(u;W)$ is real-entire in W.

We obtain the solution of eq.(2.3) as the limit $\delta \to 0$ of the solution of (2.4):

$$O_{1,m}(u;W) = O_{1,m,0}(u;W) = (\kappa_0 u)^{1/2+|m|}e^{-\rho/2}\Phi(\alpha,\beta;\rho),$$
$$O_{4,m}(u;W) = \lim_{\delta\to 0} O_{4,m,\delta}(u;W),$$
$$O_{3,m}(u;W) = (\kappa_0 u)^{1/2+|m|}e^{-\rho/2}\Psi(\alpha,\beta;\rho) =$$
$$= B_m(W)O_{1,m}(u;W) + C_m(W)O_{4,m}(u;W),\ C_m(W) = \frac{(\kappa_0/\varkappa)^{2|m|}\Gamma(|m|)}{\Gamma(\alpha)},$$
$$B_m(W) = B_{m,0}(W) = \frac{(-1)^{|m|+1}}{2\Gamma(\beta)\Gamma(\alpha_-)}\left[\psi(\alpha_-) + \psi(\alpha) - 4\ln(\kappa_0/\varkappa)\right].$$

where we used relations.

$$|m+\delta| = |m| + \delta_m,\ \delta_m = \delta\,\mathrm{sign}m,\ \Gamma(-|m+\delta|) = \frac{(-1)^{|m|+1}}{\delta_m\Gamma(\beta)},$$
$$\frac{\Gamma(\alpha_-)}{\Gamma(\alpha_{-,\delta})} = 1 + \delta_m\psi(\alpha_-)/2,\ \frac{\Gamma(\alpha)}{\Gamma(\alpha_\delta)} = 1 - \delta_m\psi(\alpha)/2,\ (\kappa_0/\varkappa)^{2\delta_m} = 1 + 2\delta_m\ln(\kappa_0/\varkappa).$$

2.3.1 Asymptotics, $u \to 0$ ($\rho \to 0$)

$$O_{1,m}(u;W) = (\kappa_0 u)^{1/2+|m|}(1 + O(u^2)), \tag{2.12}$$
$$O_{4,m}(u;W) = (\kappa_0 u)^{1/2-|m|}\left(1 + \left\{\begin{array}{c} O(u^2),\ |m| \geq 2 \\ O(u^2\ln u),\ |m| = 1 \end{array}\right.\right),$$
$$O_{3,m}(u;W) = .C_m(W)(\kappa_0 u)^{1/2-|m|}\left(1 + \left\{\begin{array}{c} O(u^2),\ |m| \geq 2 \\ O(u^2\ln u),\ |m| = 1 \end{array}\right.\right),\ \mathrm{Im}\,W > 0 \text{ or } W = 0. \tag{2.13}$$

2.3.2 Asymptotics, $u \to \infty$ ($\rho \to \infty$), $\operatorname{Im} W > 0$ or $W = 0$

$$O_{1,m}(u;W) = \frac{\kappa_0^{1/2+|m|}\varkappa^{2\alpha-2\beta}\Gamma(\beta)}{\Gamma(\alpha)}u^{-1/2-2w}e^{\rho/2}(1+O(u^{-2})),$$
$$O_{3,m}(u;W) = \kappa_0^{1/2+|m|}\varkappa^{-2\alpha}u^{-1/2+2w}e^{-\rho/2}(1+O(u^{-2})).$$

2.3.3 Wronskian

$$\operatorname{Wr}(O_{1,m}, O_{3,m}) = -2\kappa_0|m|C_m(W) = -\omega(W). \tag{2.14}$$

2.4 Symmetric operator $\hat{h}_m$

For given a differential operation $\check{h}_m$ we determine the following symmetric operator $\hat{h}_m$,

$$\hat{h}_m : \begin{cases} D_{h_m} = \mathcal{D}(\mathbb{R}_+), \\ \hat{h}_m\psi(u) = \check{h}_m\psi(u),\ \forall\psi \in D_{h_m} \end{cases}. \tag{2.15}$$

2.5 Adjoint operator $\hat{h}_m^+ = \hat{h}_m^*$

(in this file, the subindex "O" is omitted)

$$\hat{h}_m^+ : \begin{cases} D_{h_m^+} = D^*_{\check{h}_m}(\mathbb{R}_+) = \{\psi_*, \psi_*' \text{ are a.c. in } \mathbb{R}_+,\ \psi_*, \hat{h}_m^+\psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}_m^+\psi_*(u) = \check{h}_m\psi_*(u),\ \forall\psi_* \in D_{h_m^+} \end{cases}. \tag{2.16}$$

2.5.1 Asymptotics

I) $|u| \to \infty$

Because $V(u) > -(|\lambda|+1)u^2$ for large u, we have: $[\psi_*, \chi_*](u) \to 0$ as $u \to \infty$, $\forall\psi_*, \chi_* \in D_{h_m^+}$.

II) $u \to 0$

Because $\check{h}_m\psi_* \in L^2(\mathbb{R})$, we have

$$\check{h}_m\psi_*(u) = (-\partial_u^2 + u^{-2}(m^2 - 1/4) + \lambda u^2)\psi_*(u) = \eta(u),\ \eta \in L^2(\mathbb{R}).$$

General solution of this equation can be represented in the form

$$\psi_*(u) = a_1 O_{1,m}(u;0) + a_2 O_{3,m}(u;0) + I(u),$$
$$\psi_*'(u) = a_1 O_{1,m}'(u;0) + a_2 O_{3,m}'(u;0) + I'(u),$$

where

$$I(u) = \frac{O_{3,m}(u;0)}{\omega(0)}\int_0^u O_{1,m}(v;0)\eta(v)dv + \frac{O_{1,m}(u;0)}{\omega(0)}\int_u^\infty O_{3,m}(v;0)\eta(v)dv,$$
$$I'(u) = \frac{O_{3,m}'(u;0)}{\omega(0)}\int_0^u O_{1,m}(v;0)\eta(v)dv + \frac{O_{1,m}'(u;0)}{\omega(0)}\int_u^\infty O_{3,m}(v;0)\eta(v)dv$$

We obtain with the help of the Cauchy-Bunyakovskii inequality (CB-inequality): $I(u)$ is bounded at infinity and

$$I(u)=\left\{\begin{array}{l}O(u^{3/2}),\ |m|\geq 2\\ O(u^{3/2}\ln u),\ |m|=1\end{array}\right.,\ I'(u)=\left\{\begin{array}{l}O(u^{1/2}),\ |m|\geq 2\\ O(u^{1/2}\ln u),\ |m|=1\end{array}\right.,\ u\to 0.$$

The condition $\psi_*(u)\in L^2(\mathbb{R}_+)$ gives $a_1=a_2=0$ such that we find

$$\psi_*(u)=\left\{\begin{array}{l}O(u^{3/2}),\ |m|\geq 2\\ O(u^{3/2}\ln u),\ |m|=1\end{array}\right.,\ \psi_*'(u)=\left\{\begin{array}{l}O(u^{1/2}),\ |m|\geq 2\\ O(u^{1/2}\ln u),\ |m|=1\end{array}\right.,\ u\to 0, \tag{2.17}$$

and $\omega_{h_m^+}(\chi_*,\psi_*)=\Delta_{h_m^+}(\psi_*)=0$.

2.6 Self-adjoint hamiltonian $\hat{h}_{m\mathfrak{e}}$

Because $\omega_{h_m^+}(\chi_*,\psi_*)=\Delta_{h_m^+}(\psi_*)=0$ (and also because $O_{1,m}(u;W)$ and $O_{3,m}(u;W)$ and their any linear combinations are not s.-integrable for $\operatorname{Im}W\neq 0$), the deficiency indices of initial symmetric operator $\hat{h}_m$ are zero, which means that $\hat{h}_{m\mathfrak{e}}=\hat{h}_m^+$ is a unique s.a. extension of the initial symmetric operator $\hat{h}_m$:

$$\hat{h}_{m\mathfrak{e}}:\left\{\begin{array}{l}D_{h_{m\mathfrak{e}}}=D^*_{\check{h}_m}(\mathbb{R}_+)\\ \hat{h}_{m\mathfrak{e}}\psi_*(u)=\check{h}_m\psi_*(u),\ \forall\psi_*\in D_{h_{m\mathfrak{e}}}\end{array}\right.. \tag{2.18}$$

2.7 The guiding functional $\Phi(\xi;W)$

As a guiding functional $\Phi(\xi;W)$ we choose

$$\Phi(\xi;W)=\int_0^\infty O_{1,m}(u;W)\xi(u)du,\ \xi\in\mathbb{D}=D_r(\mathbb{R}_+)\cap D_{h_{m\mathfrak{e}}}. \tag{2.19}$$
$$D_r(\mathbb{R}_+)=\{\xi(u):\ \mathrm{supp}\xi\subseteq[0,\beta_\xi],\ \beta_\xi<\infty\}.$$

The guiding functional $\Phi(\xi;W)$ is simple. It has, obviously, the properties 1) and 3) and we should prove the properties 2) only (see [6], pages 245-246). Let $\Phi(\xi_0;E_0)=0$, $\xi_0\in\mathbb{D}$, $E_0\in\mathbb{R}$. As a solution $\psi(u)$ of equation

$$(\check{h}_m-E_0)\psi(u)=\xi_0(u),$$

we choose

$$\psi(u)=O_{1,m}(u;E_0)\int_u^\infty U(v)\xi_0(v)dv+U(u)\int_0^u O_{1,m}(v;E_0)\xi_0(v)dv,$$

where $U(u)$ is any solution of eq. $(\check{h}_m-E_0)U(u)=0$ satisfying the condition $\mathrm{Wr}(O_{1,m},U)=-1$. Because $\xi_0\in D_r(\mathbb{R}_+)$, the function $\psi(u)$ is well determined. Because $\xi_0\in D_r(\mathbb{R}_+)$ and $\int_0^u O_{1,m}(v;E_0)\xi_0(v)=0$ for $u>\beta_{\xi_0}$, we have $\psi(u)=0$ for $u>\beta_{\xi_0}$. Using the CB-inequality we show that $\psi(u)$ satisfies the boundary condition (2.17), that is, $\psi\in\mathbb{D}$. Thus, the guiding functional $\Phi(\xi;W)$ is simple and the spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple.

2.8 Green function $G_m(u, v; W)$, spectral function $\sigma_m(E)$

We find the Green function $G_m(u, v; W)$ as the kernel of the integral representation

$$\psi(u) = \int_0^\infty G_m(u, v; W)\eta(v)dv,\ \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{m\mathfrak{e}} - W)\psi(u) = \eta(u),\ \operatorname{Im} W > 0, \tag{2.20}$$

for $\psi \in D_{h_{m\mathfrak{e}}}$. General solution of eq. (2.20) can be represented in the form

$$\begin{aligned}
&\psi(u) = a_1 O_{1,m}(u; W) + a_3 O_{3,m}(u; W) + I(u),\\
&I(u) = \frac{O_{1,m}(u; W)}{\omega(W)} \int_u^\infty O_{3,m}(v; W)\eta(v)dv + \frac{O_{3,m}(u; W)}{\omega(W)} \int_0^u O_{1,m}(v; W)\eta(v)dv,\\
&I(u) = O\left(u^{-3/2}\right),\ u \to \infty,\ I(u) = \left\{\begin{array}{l} O\left(u^{3/2}\right),\ |m| \geq 2 \\ O\left(u^{3/2}\ln u\right),\ |m| = 1 \end{array}\right.,\ u \to 0.
\end{aligned}$$

A condition $\psi \in L^2(\mathbb{R}_+)$ gives $a_1 = a_3 = 0$, such that we find

$$\begin{aligned}
&G_m(u, v; W) = \frac{1}{\omega(W)} \left\{\begin{array}{l} O_{3,m}(u; W)O_{1,m}(v; W),\ u > v \\ O_{1,m}(u; W)O_{3,m}(v; W).\ u < v \end{array}\right. = \\
&= \Omega_m(W)O_{1,m}(u; W)O_{1,m}(v; W) + \frac{1}{2\kappa_0|m|} \left\{\begin{array}{l} O_{4,m}(u; W)O_{1,m}(v; W),\ u > v \\ O_{1,m}(u; W)O_{4,m}(v; W),\ u < v \end{array}\right., \\
&\Omega_m(W) \equiv \frac{B_m(W)}{\omega(W)} = \frac{[4\ln(\kappa_0/\varkappa) - \psi(\alpha) - \psi(\alpha_-)](1-\alpha)_{|m|}}{4\kappa_0(\kappa_0/\varkappa)^{2|m|}\Gamma^2(\beta)}.
\end{aligned} \tag{2.21}$$

Note that the last term in the r.h.s. of eq. (2.21) is real for $W = E$. From the relation

$$O_{1,m}^2(u_0; E)\sigma_m'(E) = \frac{1}{\pi} \operatorname{Im} G_m(u_0 - 0, u_0 + 0; E + i0),$$

where $f(E + i0) \equiv \lim_{\varepsilon \to +0} f(E + i\varepsilon)$, $\forall f(W)$, we find

$$\sigma_m'(E) = \frac{1}{\pi} \operatorname{Im} \Omega_m(E + i0). \tag{2.22}$$

2.9 Spectrum

2.9.1 $E \geq 0$

It is convenient to represent the function $\Omega_m(W)$ in the form

$$\begin{aligned}
&\Omega_m(W) = \Omega_{1m}(W) + \Omega_{2m}(W),\ \Omega_{1m}(W) = -\frac{\psi(\alpha)(1-\alpha)_{|m|}}{2\kappa_0(\kappa_0/\varkappa)^{2|m|}\Gamma^2(\beta)},\\
&\Omega_{2m}(W) = \frac{(\kappa_0/\varkappa)^{-2|m|}[4\ln(\kappa_0/\varkappa) + \Sigma_m(\alpha)](1-\alpha)_{|m|}}{4\kappa_0\Gamma^2(\beta)},\ \operatorname{Im} \Omega_{2m}(E) = 0,
\end{aligned}$$

where we used relation

$$\psi(\alpha_-) = \psi(\alpha) - \Sigma_m(\alpha),\ \Sigma_m(\alpha) = \sum_{l=1}^{|m|}(\alpha - l)^{-1},$$

such that we have

$$\sigma'_m(E) = -\frac{(\kappa_0/\varkappa)^{-2|m|}}{2\pi\kappa_0\Gamma^2(\beta)}\,(1-\alpha)_{|m|}\big|_{W=E}\operatorname{Im}\psi(1/2+|m|/2-(E+i0)/4\sqrt{\lambda})$$

The function $\psi(\alpha)$ is real for $W = E$ where $|\psi(\alpha)| < \infty$. Therefore, $\sigma'_m(E)$ can be not equal to zero only in the points $\psi(\alpha) = \pm\infty$, i. e., in the points $\alpha = -n$, $n \in \mathbb{Z}_+$, or

$$E_n = 2\sqrt{\lambda}[1 + |m| + 2n].$$

In the neighborhood of the points E_n we have ($W = E_n + \Delta$, $\Delta = E - E_n + i\varepsilon$, $\alpha = -n - \Delta/4\sqrt{\lambda}$)

$$\operatorname{Im}\psi(-n - \Delta/4\sqrt{\lambda}) = -4\pi\sqrt{\lambda}\delta(E - E_n),\ (1-\alpha)_{|m|}\big|_{W=E_n} = (1+n)_{|m|},$$

We thus find

$$\sigma'_m(E) = \sum_{n\in\mathbb{Z}_+} Q_n^2\delta(E - E_n),\ Q_n = \frac{(\kappa_0/\varkappa)^{-|m|}}{|m|!}\sqrt{\frac{2\sqrt{\lambda}(1+n)_{|m|}}{\kappa_0}},$$
$$\operatorname{spec}\hat{h}_{m\mathfrak{e}} = \{E_n,\ n \in \mathbb{Z}_+\}.$$

2.9.2 $E < 0$

In this case, we have $\alpha = 1/2 + |m|/2 + |E|$, the function $\Omega_{1m}(E)$ is real and $\sigma'_m(E) = 0$.

Finally: the spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple and discrete, $\operatorname{spec}\hat{h}_{m\mathfrak{e}} = \{E_n > 0,\ n \in \mathbb{Z}_+\}$, and the set of functions $\{U_{mn}(u) = Q_n O_{1,m}(u; E_n),\ n \in \mathbb{Z}_+\}$ forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

2.10 $|m| \geq 1$, $\lambda < 0$

2.11 Useful solutions

Here we again will begin with the solutions $O_{1,m,\delta}(u; W)$, $O_{2,m,\delta}(u; W)$, $O_{3,m,\delta}(u; W)$, and $O_{4,m,\delta}(u; W)$ which are given by eq. (2.6), (2.7) (2.8)-(2.11), where

$$\varkappa^2 = -i\sqrt{|\lambda|} = e^{-i\pi/2}\sqrt{|\lambda|},\ \varkappa^{-1} = e^{i\pi/4}|\lambda|^{-1/4},\ \rho = -i\sqrt{|\lambda|}u^2 =$$
$$= e^{-i\pi/2}\sqrt{|\lambda|}u^2,\ \alpha = 1/2 + |m|/2 - i\tilde{w},\ \tilde{w} \equiv \tilde{w}_O = W/(4\sqrt{|\lambda|}),$$
$$\tilde{\alpha} = 1/2 + |m|/2 + i\tilde{w} = \overline{\alpha(\overline{\tilde{w}})},$$

The functions $O_{1,m,\delta}(u; W)$, $O_{2,m,\delta}(u; W)$ and $O_{4,m,\delta}(u; W)$ are real-entire in W. Indeed, we have

$$\overline{O_{1,m,\delta}(u; W)} = (\kappa_0 u)^{1/2+|m+\delta|}e^{\rho/2}\Phi(\alpha_\delta|_{W\to-\overline{W}}, \beta_\delta; -\rho) =$$
$$= (\kappa_0 u)^{1/2+|m+\delta|}e^{-\rho/2}\Phi(\alpha_\delta|_{W\to\overline{W}}, \beta_\delta; \rho) = O_{1,m,\delta}(u; \overline{W}),$$

where we used the identity GradRy9.212.1 Analogously, we find

$$\overline{O_{2,m,\delta}(u;W)} = O_{2,m,\delta}(u;\overline{W}).$$

The real-entireness of $O_{4,m,\delta}(u;W)$ follows from the fact that $O_{1,m,\delta}(u;W)$, $O_{2,m,\delta}(u;W)$, and $A_{m,\delta}(W)$ are real-entire in W.

2.11.1 Asymptotics, $u\to 0$ $(\rho\to 0)$

Asymptotics of $O_{1,m,\delta}(u;W)$ and $O_{3,m,\delta}(u;W)$ are given by eqs. (2.12) and (2.13).

2.11.2 Asymptotics, $u\to\infty$ $(\rho\to -i\infty)$, $\operatorname{Im} W>0$ or $W=0$

$$\begin{aligned}
&O_{1,m}(u;W) = \frac{\kappa_0^{1/2+|m|}\varkappa^{2\alpha-2\beta}\Gamma(\beta)}{\Gamma(\alpha)}u^{-1/2-2w}e^{\rho/2}(1+O(u^{-2})) = \\
&= O(u^{-1/2+\operatorname{Im} W/2\sqrt{|\lambda|}}),\\
&O_{3,m}(u;W) = \kappa_0^{1/2+|m|}\varkappa^{-2\alpha}u^{-1/2+2w}e^{-\rho/2}(1+O(u^{-2})) = \\
&= O(u^{-1/2-\operatorname{Im} W/2\sqrt{|\lambda|}}).
\end{aligned}$$

-

2.11.3 Wronskian

The Wronskian of $O_{1,m,\delta}(u;W)$ and $O_{3,m,\delta}(u;W)$ is given by eq. (2.14).

2.12 Symmetric operator $\hat{h}_m$, adjoint operator $\hat{h}_m^+ = \hat{h}_m^*$, s.a. hamiltonian $\hat{h}_{m\mathfrak{e}}$, guiding functional $\Phi(\xi;W)$, Green function $G_m(u,v;W)$, spectral function $\sigma_m(E)$

These quantities are given exactly by the same formulas (2.15), (2.16), (2.18), (2.19), (2.21), and (2.22) for the corresponding quantities of previous section.

Let us study the structure of exp.(2.22) for $\sigma_m'(E)$.

Let $|m|=2n+1$, $n\in\mathbb{Z}_+$. We have

$$\begin{aligned}
&(\kappa_0/\varkappa)^{-2|m|} = (-i)^{|m|}(\sqrt{|\lambda|}/\kappa_0^2)^{|m|},\ (1-\alpha)_{|m|} = i^{|m|}\tilde{e}\prod_{l=1}^{n}(l^2+\tilde{e}^2),\\
&(\kappa_0/\varkappa)^{-2|m|}(1-\alpha)_{|m|} = \tilde{e}q_{1n}(E),\ q_{1n}(E) = (\sqrt{|\lambda|}/\kappa_0^2)^{|m|}\prod_{l=1}^{n}(l^2+\tilde{e}^2) > 0,\\
&\operatorname{Im}\psi(\alpha)+\operatorname{Im}\psi(\alpha_-) = \operatorname{Im}\psi(1+n-i\tilde{e})+\operatorname{Im}\psi(-n-i\tilde{e}) = -\pi\coth(\pi\tilde{e}),\\
&\ln(\kappa_0/\varkappa) = i\pi/4+\ln(\kappa_0|\lambda|^{-1/4}),\ \tilde{e} = E/4\sqrt{|\lambda|},
\end{aligned}$$

and

$$\sigma_m'(E) \equiv \rho_m^2(E) = \frac{\tilde{e}q_{1n}(E)}{4\kappa_0\Gamma^2(\beta)}[\coth(\pi\tilde{e})+1] > 0.$$

Let $|m| = 2n$, $n \in \mathbb{N}$. We have

$$
\begin{aligned}
&(\kappa_0/\varkappa)^{-2|m|} = (-i)^{|m|}(\sqrt{|\lambda|}/\kappa_0^2)^{|m|},\ (1-\alpha)_{|m|} = i^{|m|}\prod_{l=0}^{n-1}[(l+1/2)^2+\tilde{e}^2],\\
&(\kappa_0/\varkappa)^{-2|m|}(1-\alpha)_{|m|} = q_{2n}(E),\ q_{2n}(E) = (\sqrt{|\lambda|}/\kappa_0^2)^{|m|}\prod_{l=0}^{n-1}[(l+1/2)^2+\tilde{e}^2] > 0,\\
&\operatorname{Im}\psi(\alpha) + \operatorname{Im}\psi(\alpha_-) = \operatorname{Im}\psi(1/2+n-i\tilde{e}) + \operatorname{Im}\psi(1/2-n-i\tilde{e}) = -\pi\tanh(\pi\tilde{e}),\\
&\ln(\kappa_0/\varkappa) = i\pi/4 + \ln(\kappa_0|\lambda|^{-1/4}),
\end{aligned}
$$

and

$$
\sigma'_m(E) \equiv \rho_m^2(E) = \frac{q_{2n}(E)}{4\kappa_0\Gamma^2(\beta)}[1+\tanh(\pi\tilde{e})] > 0.
$$

The function $\sigma'_m(E)$ is absolutely continuous for any $E \in \mathbb{R}$, such that we obtain: the spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple, continuous and it fills out real axis, $\operatorname{spec}\hat{h}_{m\mathfrak{e}} = \mathbb{R}$.The set of (generalized) eigenfunctions $\{U_{mE}(u) = \rho_m(E)O_{1,m}(u;E),\ E \in \mathbb{R}\}$ of $\hat{h}_{m\mathfrak{e}}$ forms a complete orthonormalized system.

2.13 $|m| \geq 1$, $\lambda = 0$

2.14 Useful solutions

Here we set $\delta = 0$ at once. Eq. (2.3) is reduced to the form

$$
[-\partial_u^2 + u^{-2}(m^2 - 1/4) - W]\psi(u) = 0. \tag{2.23}
$$

The solutions of eq. (2.23) are expressed in the terms of the Bessel functions. We use the following solutions:

$$
\begin{aligned}
&O_{1,m}(u;W) = D_{1,m}(W)u^{1/2}J_{|m|}(Ku),\ D_{1,m}(W) = \kappa_0^{1/2}\Gamma(\beta)(K/2\kappa_0)^{-|m|},\\
&O_{3,m}(u;W) = iD_{3,m}(W)u^{1/2}H_{|m|}^{(1)}(Ku),\ D_{3,m}(W) = \pi\kappa_0^{1/2}\Gamma^{-1}(\beta-1)(K/2\kappa_0)^{|m|},\\
&O_{4,m}(u;W) = D_{3,m}(W)u^{1/2}\left[N_{|m|}(Ku) - \frac{2}{\pi}J_{|m|}(Ku)\ln(K/\kappa_0)\right],\\
&O_{3,m}(u;W) = \pi|m|\omega_m(W)\left[i - \frac{2}{\pi}\ln(K/\kappa_0)\right]O_{1,m}(u;W) - O_{4,m}(u;W),\\
&K = W^{1/2} = \sqrt{|W|}e^{i\varphi_W/2},\ \omega_m(W) = \frac{D_{3,m}(W)}{\pi|m|D_{1,m}(W)} = \frac{(W/4\kappa_0^2)^{|m|}}{\Gamma^2(\beta)}.
\end{aligned}
$$

2.14.1 Asymptotics, $u \to 0$ $(z \to 0)$

$$
\begin{aligned}
&O_{1,m}(u;W) = (\kappa_0 u)^{1/2+|m|}(1+O(u^2)),\\
&O_{4,m}(u;W) = -(\kappa_0 u)^{1/2-|m|}\left(1 + \left\{\begin{array}{c} O(u^2),\ |m| \geq 2 \\ O(u^2\ln u),\ |m| = 1 \end{array}\right.\right),\\
&O_{3,m}(u;W) = (\kappa_0 u)^{1/2-|m|}\left(1 + \left\{\begin{array}{c} O(u^2),\ |m| \geq 2 \\ O(u^2\ln u),\ |m| = 1 \end{array}\right.\right).
\end{aligned}
$$

2.14.2 Asymptotics, $u \to \infty$ ($z \to \infty$), $\operatorname{Im} W > 0$ ($\operatorname{Im} K > 0$)

$$
\begin{aligned}
O_{1,m}(u;W) &= D_{1,m}(W)\sqrt{\frac{1}{2\pi K}} e^{i\pi(|m|/2+1/4)} e^{-iKu}(1+O(u^{-1})) = O(e^{u \operatorname{Im} K}),\\
O_{3,m}(u;W) &= iD_{3,m}(W)\sqrt{\frac{2}{\pi K}} e^{-i\pi(|m|/2+1/4)} e^{iKu}(1+O(u^{-1})). = O(e^{-u \operatorname{Im} K}).
\end{aligned}
$$

2.14.3 Wronskian

$$
\operatorname{Wr}(O_{1,m}, O_{3,m}) = -2\kappa_0 |m|.
$$

2.15 Symmetric operator $\hat{h}_m$

For given a differential operation $\check{h}_m = -\partial_u^2 + u^{-2}(m^2 - 1/4)$ we determine the following symmetric operator $\hat{h}_m$,

$$
\hat{h}_m : \left\{ \begin{array}{l} D_{h_m} = \mathcal{D}(\mathbb{R}_+), \\ \hat{h}_m \psi(u) = \check{h}_m \psi(u), \ \forall \psi \in D_{h_m} \end{array} \right. .
$$

2.16 Adjoint operator $\hat{h}_m^+ = \hat{h}_m^*$

$$
\hat{h}_m^+ : \left\{ \begin{array}{l} D_{h_m^+} = D^*_{\check{h}_m}(\mathbb{R}_+) = \{\psi_*, \psi_*' \text{ are a.c. in } \mathbb{R}_+, \ \psi_*, \hat{h}_m^+ \psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}_m^+ \psi_*(u) = \check{h}_m \psi_*(u), \ \forall \psi_* \in D_{h_m^+} \end{array} \right. .
$$

2.16.1 Asymptotics

I) $|u| \to \infty$

Because $V(u) > -(|\lambda| + 1)u^2$ for large u, we have: $[\psi_*, \chi_*](u) \to 0$ as $u \to \infty$, $\forall \psi_*, \chi_* \in D_{H^+}$.

II) $u \to 0$

Because $\check{h}_m \psi_* \in L^2(\mathbb{R})$, we have

$$
\check{h}_m \psi_*(u) = [-\partial_u^2 + u^{-2}(m^2 - 1/4)]\psi_*(u) = \eta(u), \ \eta \in L^2(\mathbb{R}_+).
$$

General solution of this equation can be represented in the form

$$
\begin{aligned}
\psi_*(u) &= a_1 u^{1/2+|m|} + a_2 u^{1/2-|m|} + I(u),\\
\psi_*'(u) &= (a_1 u^{1/2+|m|} + a_2 u^{1/2-|m|})' + I'(u),
\end{aligned}
$$

where

$$
\begin{aligned}
I(u) &= \frac{u^{1/2-|m|}}{2|m|} \int_0^u v^{1/2+|m|} \eta(v) dv + \frac{u^{1/2+|m|}}{2|m|} \int_u^{x_0} v^{1/2-|m|} \eta(v) dv,\\
I'(u) &= \frac{(u^{1/2-|m|})'}{2|m|} \int_0^u v^{1/2+|m|} \eta(v) dv + \frac{(u^{1/2+|m|})'}{2|m|} \int_u^{x_0} v^{1/2-|m|} \eta(v) dv
\end{aligned}
$$

We obtain with the help of the Cauchy-Bunyakovskii inequality (CB-inequality): $I(u)$ is bounded at infinity and

$$I(u)=\left\{\begin{array}{l} O(u^{3/2}),\ |m|\geq 2 \\ O(u^{3/2}\ln u),\ |m|=1 \end{array}\right.,\ I'(u)=\left\{\begin{array}{l} O(u^{1/2}),\ |m|\geq 2 \\ O(u^{1/2}\ln u),\ |m|=1 \end{array}\right.,\ u\to 0.$$

The condition $\psi_*(u)\in L^2(\mathbb{R}_+)$ gives $a_1=0$, such that we find

$$\psi_*(u)=\left\{\begin{array}{l} O(u^{3/2}),\ |m|\geq 2 \\ O(u^{3/2}\ln u),\ |m|=1 \end{array}\right.,\ \psi_*'(u)=\left\{\begin{array}{l} O(u^{1/2}),\ |m|\geq 2 \\ O(u^{1/2}\ln u),\ |m|=1 \end{array}\right.,\ u\to 0,$$

and $\omega_{h_m^+}(\chi_*,\psi_*)=\Delta_{h_m^+}(\psi_*)=0$

2.17 Self-adjoint hamiltonian $\hat{h}_{m\mathfrak{e}}$

Because $\omega_{h_m^+}(\chi_*,\psi_*)=\Delta_{h_m^+}(\psi_*)=0$ (and also because $O_{1,m}(u;W)$ and $O_{3,m}(u;W)$ and their linear combinations are not s.-integrable for $\operatorname{Im}W\neq 0$), the deficiency indices of initial symmetric operator $\hat{h}_m$ are zero, which means that $\hat{h}_{m\mathfrak{e}}=\hat{h}_m^+$ is a unique s.a. extension of the initial symmetric operator $\hat{h}_m$:

$$\hat{h}_{m\mathfrak{e}}:\left\{\begin{array}{l} D_{h_{m\mathfrak{e}}}=D^*_{\check{h}_m}(\mathbb{R}_+) \\ \hat{h}_{m\mathfrak{e}}\psi_*(u)=\check{h}_m\psi_*(u),\ \forall\psi_*\in D_{h_{m\mathfrak{e}}} \end{array}\right. .$$

2.18 The guiding functional $\Phi(\xi;W)$

As a guiding functional $\Phi(\xi;W)$ we choose

$$\begin{aligned} &\Phi(\xi;W)=\int_0^\infty O_{1,m}(u;W)\xi(u)du,\ \xi\in\mathbb{D}_\zeta=D_r(\mathbb{R}_+)\cap D_{h_{m\mathfrak{e}}}. \\ &D_r(a,b)=\{\psi(u):\ \operatorname{supp}\psi\subseteq[a,\beta_\psi],\ \beta_\psi<b. \end{aligned}$$

The guiding functional $\Phi_\zeta(\xi;W)$ is simple. and the spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple.

2.19 Green function $G_m(u,v;W)$, spectral function $\sigma_m(E)$

We find the Green function $G_m(u,v;W)$ as the kernel of the integral representation

$$\psi(u)=\int_0^\infty G_m(u,v;W)\eta(v)dv,\ \eta\in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{m\mathfrak{e}}-W)\psi(u)=\eta(u),\ \operatorname{Im}W>0, \tag{2.24}$$

for $\psi\in D_{h_{m\mathfrak{e}}}$. General solution of eq. (2.24) can be represented in the form

$$\begin{aligned} &\psi(u)=a_1O_{1,m}(u;W)+a_3O_{3,m}(u;W)+I(u), \\ &I(u)=\frac{O_{1,m}(u;W)}{2\kappa_0|m|}\int_u^\infty O_{3,m}(v;W)\eta(v)dv+\frac{O_{3,m}(u;W)}{2\kappa_0|m|}\int_0^u O_{1,m}(v;W)\eta(v)dv, \\ &I(u)=O\left(u^{-3/2}\right),\ u\to\infty,\ I(u)=\left\{\begin{array}{l} O\left(u^{3/2}\right),\ |m|\geq 2 \\ O\left(u^{3/2}\ln u\right),\ |m|=1 \end{array}\right.,\ u\to 0. \end{aligned}$$

A condition $\psi \in L^2(\mathbb{R}_+)$ gives $a_1 = a_3 = 0$, such that we find

$$G_m(u,v;W) = \frac{1}{2\kappa_0|m|}\left\{\begin{array}{l} O_{3,m}(u;W)O_{1,m}(v;W),\ u>v \\ O_{1,m}(u;W)O_{3,m}(v;W).\ u<v \end{array}\right. =$$

$$= \Omega_m(W)O_{1,m}(u;W)O_{1,m}(v;W) - \frac{1}{2\kappa_0|m|}\left\{\begin{array}{l} O_{4,m}(u;W)O_{1,m}(v;W),\ u>v \\ O_{1,m}(u;W)O_{4,m}(v;W),\ u<v \end{array}\right., \qquad (2.25)$$

$$\Omega_m(W) \equiv \frac{\pi}{2\kappa_0}\omega_m(W)[i-(2/\pi)\ln(K/\kappa_0)] = \frac{\pi(W/4\kappa_0^2)^{|m|}}{2\kappa_0\Gamma^2(\beta)}[i-(2/\pi)\ln(K/\kappa_0)].$$

Note that the last term in the r.h.s. of eq. (2.25) is real for $W = E$. We find

$$\sigma_m'(E) = \frac{1}{\pi}\operatorname{Im}\Omega_m(E+i0).$$

2.20 Spectrum

2.20.1 $E = p^2 \geq 0$ $(K = p)$

$$\sigma_m'(E) = \rho_m^2(E),\ \rho_m(E) = \frac{(p/2\kappa_0)^{|m|}}{\sqrt{2\kappa_0}|m|!}$$

The spectrum is simple and continuous on whole nonnegative E-semiaxis.

2.20.2 $E = -\tau^2 < 0$

In this case, we have $K = e^{i\pi/2}\tau$, the function $\Omega_m(E)$ is real and $\sigma_m'(E) = 0$.

Finally: the spectrum $\hat{h}_{m\mathfrak{e}}$ is simple and continuous, spec$\hat{h}_{m\mathfrak{e}} = \{E \geq 0\}$, and the set of functions $\{U_{mE}(u),\ E \geq 0\}$,

$$U_{mE}(u) = \rho_m(E)O_{1,m}(u;E) = \sqrt{u/2}J_{|m|}(pu),$$

forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

Note that the results of this section can be obtained as limit $\lambda \to 0$ of the corresponding results of previous sec.3.

Indeed, we have

$$\lim_{\lambda\to 0} O_{1,m}(u;W)|_{\lambda\neq 0} = (\kappa_0 u)^{1/2+|m|}\sum_{k=0}^{\infty}\frac{(-W/4\varkappa^2)^k\Gamma(\beta)}{\Gamma(\beta+k)}\frac{\varkappa^{2k}u^{2k}}{k!} =$$

$$= \kappa_0^{1/2}(K/2\kappa_0)^{-|m|}\Gamma(\beta)\left[(Ku/2)^{|m|}\sum_{k=0}^{\infty}\frac{(-1)^k(Ku/2)^{2k}}{\Gamma(\beta+k)k!}\right] =$$

$$= D_{1,m}(W)u^{1/2}J_{|m|}(Ku) = O_{1,m}(u;W)|_{\lambda=0}.$$

Further, we have for $\sigma_m'(E)$

i) $E > 0$

$$\lim_{\lambda\to -0}\sigma_m'(E)|_{\lambda<0} = \frac{1}{2\kappa_0\Gamma^2(\beta)}\left(\frac{E}{4\kappa_0^2}\right)^{|m|}$$

ii) $E < 0$

$$\lim_{\lambda\to -0}\sigma_m'(E)|_{\lambda<0} = 0.$$

iii) $E = 0$

$$\sigma_m'(0)|_{\lambda<0} = 0.$$

2.21 $m = 0$

2.22 Useful solutions

For $m = 0$, eqs. (2.3) and (2.4) are redused respectively to

$$(-\partial_u^2 - u^{-2}/4 + \lambda u^2 - W)\psi(u) = 0 \tag{2.26}$$

and

$$[-\partial_u^2 + u^{-2}(\delta^2 - 1/4) + \lambda u^2 - W]\psi(u) = 0.$$

We will use the following solutions of eq. (2.26)

$$\begin{aligned}
&O_{1,0}(u;W) = O_{1,0,0}(u;W) = (\kappa_0 u)^{1/2} e^{-\rho/2}\Phi(\alpha, 1; \rho),\\
&O_{2,0}(u;W) = \partial_\delta O_{1,0,\delta}(u;W)|_{\delta=+0} =\\
&\ = (\kappa_0 u)^{1/2} e^{-\rho/2}\ \partial_\delta \Phi(\alpha_\delta, \beta_\delta; \rho)|_{\delta=+0} + O_{1,0}(u;W)\ln(\kappa_0 u),\\
&O_{3,0}(u;W) = O_{3,0,0}(u;W) = (\kappa_0 u)^{1/2} e^{-\rho/2}\Psi(\alpha, 1; \rho) =\\
&\ = \frac{\omega_0(W)}{\Gamma(\alpha)} O_{1,0}(u;W) - \frac{2}{\Gamma(\alpha)} O_{2,0}(u;W),\\
&\alpha = 1/2 - w,\ \omega_0(W) = 2\ln(\kappa_0/\varkappa) + 2\psi(1) - \psi(\alpha).
\end{aligned}$$

2.22.1 Asymptotics, $u \to 0$

We have

$$\begin{aligned}
&O_{1,0}(u;W) = (\kappa_0 u)^{1/2}(1 + O(u^2)),\ O_{2,0}(u;W) = (\kappa_0 u)^{1/2}\ln(\kappa_0 u)(1 + O(u^2)),\\
&O_{3,0}(u;W) = \left[\frac{\omega_0(W)}{\Gamma(\alpha)}(\kappa_0 u)^{1/2} - \frac{2}{\Gamma(\alpha)}(\kappa_0 u)^{1/2}\ln(\kappa_0 u)\right](1 + O(u^2)).
\end{aligned}$$

2.22.2 Wronskian

$$\mathrm{Wr}(O_{1,0}, O_{3,0}) = -\frac{2\kappa_0}{\Gamma(\alpha)}$$

2.23 $\lambda > 0$ $(\varkappa = \lambda^{1/4})$

2.23.1 Asymptotics, $u \to \infty$, $\operatorname{Im} W > 0$ or $W = 0$

We have

$$\begin{aligned}
&O_{1,0}(u;W) = \frac{\kappa_0^{1/2}\varkappa^{-1-2w}}{\Gamma(\alpha)} u^{-1/2-2w} e^{\rho/2}(1 + O(u^{-2})),\\
&O_{3,0}(u;W) = \kappa_0^{1/2}\varkappa^{-1+2w} u^{-1/2+2w} e^{-\rho/2}(1 + O(u^{-2})).
\end{aligned}$$

2.23.2 Symmetric operator $\hat{h}_0$

For given a differential operation $\check{h}_0 = -\partial_u^2 - u^{-2}/4 + \lambda u^2$ we determine the following symmetric operator $\hat{h}_0$,

$$\hat{h}_0 : \left\{ \begin{array}{l} D_{h_0} = \mathcal{D}(\mathbb{R}_+),\\ \hat{h}_0\psi(u) = \check{h}_0\psi(u),\ \forall\psi \in D_{h_0} \end{array} \right. . \tag{2.27}$$

2.23.3 Adjoint operator $\hat{h}_0^+ = \hat{h}_0^*$

$$\hat{h}_0^+ : \begin{cases} D_{h_0^+} = D^*_{\check{h}_0}(\mathbb{R}_+) = \{\psi_*, \psi'_* \text{ are a.c. in } \mathbb{R}_+,\ \psi_*, \hat{h}_0^+\psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}_0^+\psi_*(u) = \check{h}_0\psi_*(u),\ \forall\psi_* \in D_{h_0^+} \end{cases} . \tag{2.28}$$

Asymptotics I) $u` \to \infty$

Because $V(u) > -(|\lambda| + 1)u^2$ for large u, we have: $[\psi_*, \chi_*](u) \to 0$ as $u \to \infty$, $\forall\psi_*, \chi_* \in D_{H^+}$.

II) $u \to 0$

By the standard way, we obtain

$$\begin{aligned} &\psi_*(u) = c_1 u_{1\text{as}}(u) + c_2 u_{2\text{as}}(u) + O(u^{3/2}\ln u), \\ &\psi'_*(u) = c_1 u'_{1\text{as}}(u) + c_2 u'_{2\text{as}}(u) + O(u^{1/2}\ln u), \\ &u_{1\text{as}}(u) = (\kappa_0 u)^{1/2},\ u_{2\text{as}}(u) = (\kappa_0 u)^{1/2}\ln(\kappa_0 u). \end{aligned} \tag{2.29}$$

For the asymmetry form $\Delta_{h_0^+}(\psi_*)$, we find

$$\begin{aligned} &\Delta_{h_0^+}(\psi_*) = \kappa_0(\overline{c_2}c_1 - \overline{c_1}c_2) = i\kappa_0(\overline{c_+}c_+ - \overline{c_-}c_-), \\ &c_\pm = \frac{1}{\sqrt{2}}(c_1 \pm ic_2). \end{aligned} \tag{2.30}$$

2.23.4 Self-adjoint hamiltonians

The condition $\Delta_{h_0^+}(\psi) = 0$ gives

$$\begin{aligned} &c_- = e^{2i\theta}c_+,\ 0 \le \theta \le \pi,\ \theta = 0 \sim \theta = \pi,\ \Longrightarrow \\ &c_1\cos\zeta = c_2\sin\zeta,\ \zeta = \theta - \pi/2,\ |\zeta| \le \pi/2,\ \zeta = -\pi/2 \sim \zeta = \pi/2, \end{aligned}$$

or

$$\begin{aligned} &\psi(u) = C\psi_{\text{as}}(u) + O(u^{3/2}\ln u),\ \psi'(u) = C\psi'_{\text{as}}(u) + O(u^{1/2}\ln u), \\ &\psi_{\text{as}}(u) = u_{1\text{as}}(u)\sin\zeta + u_{2\text{as}}(u)\cos\zeta. \end{aligned} \tag{2.31}$$

We thus have a family of s.a. $\hat{h}_{0\zeta}$,

$$\hat{h}_{0\zeta} : \begin{cases} D_{h_{0\zeta}} - \{\psi \in D_{h_0^+},\ \psi \text{ satisfy the boundary condition (2.31)} \\ \hat{h}_{0\zeta}\psi = \check{h}_0\psi,\ \forall\psi \in D_{h_{0\zeta}} \end{cases} . \tag{2.32}$$

2.23.5 The guiding functional

As a guiding functional $\Phi_\zeta(\xi; W)$ we choose

$$\begin{aligned} &\Phi_\zeta(\xi; W) = \int_0^\infty U_\zeta(u; W)\xi(u)du,\ \xi \in \mathbb{D}_\zeta = D_r(\mathbb{R}_+) \cap D_{h_{0\zeta}}. \\ &.U_\zeta(u; W) = O_{1,0}(u; W)\sin\zeta + O_{2,0}(u; W)\cos\zeta, \end{aligned} \tag{2.33}$$

$U_\zeta(u; W)$ is real-entire solution of eq. (2.26) satisfying the boundary condition (2.31).

The guiding functional $\Phi_\zeta(\xi; W)$ is simple and the spectrum of $\hat{h}_{0\zeta}$ is simple.

2.23.6 Green function $G_\zeta(u,v;W)$, spectral function $\sigma_\zeta(E)$.

We find the Green function $G_\zeta(u,v;W)$ as the kernel of the integral representation

$$\psi(u) = \int_0^\infty G_\zeta(u,v;W)\eta(v)dv,\ \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{0\zeta} - W)\psi(u) = \eta(u),\ \operatorname{Im} W > 0, \tag{2.34}$$

for $\psi \in D_{h_{0\zeta}}$. General solution of eq. (2.34) (under condition $\psi \in L^2(\mathbb{R}_+)$) can be represented in the form

$$\begin{aligned}
&\psi(u) = aO_{3,0}(u;W) + \frac{\Gamma(\alpha)}{2\kappa_0}O_{1,0}(u;W)\eta_3(W) + \frac{\Gamma(\alpha)}{2\kappa_0}I(u),\ \eta_3(W) = \int_0^\infty O_{3,0}(v;W)\eta(v)dv,\\
&I(u) = O_{3,0}(u;W)\int_0^u O_{1,0}(v;W)\eta(v)dv - O_{1,0}(u;W)\int_0^u O_{3,0}(v;W)\eta(v)dv,\\
&I(u) = O\left(u^{3/2}\ln u\right),\ u \to 0.
\end{aligned}$$

A condition $\psi \in D_{h_{0\zeta}}$, (i.e.ψ satisfies the boundary condition (2.31)) gives

$$a = -\frac{\Gamma^2(\alpha)\cos\zeta}{4\kappa_0\omega_\zeta(W)}\eta_3(W),\ \omega_\zeta(W) = \frac{1}{2}\omega_0(W)\cos\zeta + \sin\zeta,$$

$$\begin{aligned}
&G_\zeta(u,v;W) = \Omega_\zeta(W)U_\zeta(u;W)U_\zeta(v;W)+\\
&+\frac{1}{\kappa_0}\begin{cases} \tilde{U}_\zeta(u;W)U_\zeta(v;W),\ u>v \\ U_\zeta(u;W)\tilde{U}_\zeta(v;W),\ u<v \end{cases},\\
&\Omega_\zeta(W) \equiv \frac{\tilde{\omega}_\zeta(W)}{\kappa_0\omega_\zeta(W)},\ \tilde{\omega}_\zeta(W) = \frac{1}{2}\omega_0(W)\sin\zeta - \cos\zeta,\\
&\tilde{U}_\zeta(u;W) = O_{1,0}(u;W)\cos\zeta - O_{2,0}(u;W)\sin\zeta,
\end{aligned} \tag{2.35}$$

where we used an equality

$$\Gamma(\alpha)O_{3,0}(u;W) = 2\tilde{\omega}_\zeta(W)U_\zeta(u;W) + 2\omega_\zeta(W)\tilde{U}_\zeta(u;W).$$

Note that the function $\tilde{U}_\zeta(u;W)$ is real-entire in W and the last term in the r.h.s. of eq. (2.35) is real for $W=E$. For $\sigma'_\zeta(E)$, we find

$$\sigma'_\zeta(E) = \frac{1}{\pi}\operatorname{Im}\Omega_\zeta(E+i0). \tag{2.36}$$

2.23.7 Spectrum

$\zeta = \pi/2$ First we consider the case $\zeta = \pi/2$.

In this case, we have $U_{\pi/2}(u;W) = O_{1,0}(u;W)$ and

$$\sigma'_{\pi/2}(E) = -\frac{1}{2\pi\kappa_0}\operatorname{Im}\psi(1/2-(E+i0)/4\sqrt{\lambda}),$$

$E \geq 0$ In this case, we find

$$\sigma'_{\pi/2}(E) = \sum_{n=0}^{\infty} \frac{2\sqrt{\lambda}}{\kappa_0}\delta(E - \mathcal{E}_n),\ \mathcal{E}_n = 2\sqrt{\lambda}(1+2n),$$
$$\text{spec}\hat{h}_{0\pi/2} = \{\mathcal{E}_n,\ n \in \mathbb{Z}_+\}..$$

$E < 0$ In this case, we have $\alpha = 1/2 + |E|/4\sqrt{\lambda}$, the function $\omega_0(E)$ is real and $\sigma'_0(E) = 0$.

Finally: the spectrum $\hat{h}_{\pi/2}$ is simple and discrete, $\text{spec}\hat{h}_{\pi/2} = \{\mathcal{E}_n > 0,\ n \in \mathbb{Z}_+\}$, and the set of eigenfunctions $\{U_n(u) = (2\sqrt{\lambda}/\kappa_0)^{1/2} O_{1,0}(u; \mathcal{E}_n),\ n \in \mathbb{Z}_+\}$ forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

Note that these results for spectrum and the set of eigenfunctions can be obtained from the corresponding results of sec. 2 by formal substitution $|m| \to 0$.

The same results we obtain for the case $\zeta = -\pi/2$.

$|\zeta| < \pi/2$ Now we consider the case $|\zeta| < \pi/2$.

In this case, we can represent $\sigma'_\zeta(E)$ in the form

$$\sigma'_\zeta(E) = -\frac{1}{\pi\kappa_0 \cos^2\zeta}\,\text{Im}\,\frac{1}{f_\zeta(E+i0)},\ f_\zeta(W) = f(W) + \tan\zeta,$$
$$f(W) = \omega_0(W)/2,\ f'(E) = 4\sqrt{\lambda}\partial_\alpha\psi(\alpha)/8\sqrt{\lambda} > 0,\ \alpha \neq -n,\ n \in \mathbb{Z}_+.$$

The function $f(E)$ has the properties: $f(E) \to -\infty$ as $E \to \mathcal{E}_{-1} \equiv -\infty$; $f(\mathcal{E}_n \pm 0) = \mp\infty$, $n \in \mathbb{Z}_+$; in any interval $(\mathcal{E}_{n-1}, \mathcal{E}_n)$, $n \in \mathbb{Z}_+$, $f(E)$ increases monotonically from $-\infty$ to ∞ as E run from $\mathcal{E}_{n-1}$ to $\mathcal{E}_n$.

The function $f_\zeta^{-1}(E)$ is real in the points where $f_\zeta(E) \neq 0$ therefore $\sigma'_\zeta(E)$ can be not equal to zero only in the points $E_n(\zeta)$ satisfying the equation $f_\zeta(E) = 0$ or

$$f(E_n(\zeta)) = -\tan\zeta,\ \partial_\zeta E_n(\zeta) = -\frac{1}{f'(E_n(\zeta))\cos^2\zeta} < 0.. \tag{2.37}$$

The described above properties of function $f(E)$ imply the following structure of discrete spectrum: in each interval energy $(\mathcal{E}_{n-1}, \mathcal{E}_n)$, $n \in \mathbb{Z}_+$, for fixed $\zeta \in (-\pi/2, \pi/2)$, exists only one solution $E_n(\zeta)$ of eq. (2.37) monotonically increasing from $\mathcal{E}_{n-1} + 0$ to $\mathcal{E}_n - 0$ as ζ run from $\pi/2$ to $-\pi/2$. Note the relation

$$\lim_{\zeta\to-\pi/2} E_n(\zeta) = \lim_{\zeta\to\pi/2} E_{n+1}(\zeta) = \mathcal{E}_n,\ n \in \mathbb{Z}_+.$$

Finally we obtain:

$$\sigma'_\zeta(E) = \sum_{n\in\mathbb{Z}_+} Q_n^2\delta(E - E_n(\zeta)),\ Q_n = \frac{1}{\cos\zeta}\frac{1}{\kappa_0\sqrt{\kappa_0 f'(E_n(\zeta))}} > 0,$$

the spectrum of $\hat{h}_{0\zeta}$ is simple and discrete, $\text{spec}\hat{h}_{0\zeta} = \{E_n(\zeta),\ n \in \mathbb{Z}_+\}$, the set $\{U_n(u) = Q_n U_\zeta(u; E_n(\zeta)\},\ n \in \mathbb{Z}_+\}$ of eigenfunctions of $\hat{h}_{0\zeta}$ forms the complete orthohonalized system in $L^2(\mathbb{R}_+)$.

2.24 $\lambda < 0$ ($\varkappa = e^{-i\pi/4}|\lambda|^{1/4}$, $\rho = e^{-i\pi/2}|\lambda|^{1/2}u^2$, $w = i\tilde{e}$)

2.24.1 Asymptotics, $u \to \infty$, $\operatorname{Im} W > 0$ or $W = 0$

We have

$$O_{1,0}(u;W) = \frac{\kappa_0^{1/2}\varkappa^{-1-2i\tilde{e}}}{\Gamma(\alpha)} u^{-1/2-2i\tilde{e}} e^{i|\lambda|^{1/2}u^2/2}(1 + O(u^{-2})) = O(u^{-1/2+\operatorname{Im} W/2\sqrt{|\lambda|}},$$
$$O_{3,0}(u;W) = \kappa_0^{1/2}\varkappa^{-1+2i\tilde{e}} u^{-1/2+2i\tilde{e}} e^{i|\lambda|^{1/2}u^2/2}(1 + O(u^{-2})) = O(u^{-1/2-\operatorname{Im} W/2\sqrt{|\lambda|}}).$$

2.24.2 Symmetric operator $\hat{h}_0$, adjoint operator $\hat{h}_0^+ = \hat{h}_0^*$

Symmetric operator $\hat{h}_0$ is given by eq. (2.27), adjoint operator $\hat{h}_0^+ = \hat{h}_0^*$ is given by eqs. (2.28) – (2.30)

2.24.3 Self-adjoint hamiltonians $\hat{h}_{0\vartheta}$, guiding functional

There are a family of s.a. hamiltonians $\hat{h}_{0\vartheta}$ given by eqs. (2.32), (2.31) (with substitution $\zeta \to \vartheta$), a simple guiding functional $\Phi_\vartheta(\xi;W)$ is given by eq. (2.33) (with substitution $\zeta \to \vartheta$). The spectra of the s.a. hamiltonians $\hat{h}_{0\zeta}$ are simple.

2.24.4 Green function $G_\vartheta(u,v;W)$, spectral function $\sigma_\vartheta(E)$

The Green function $G_\vartheta(u,v;W)$ is given by eq. (2.35) and the derivative of the spectrum function $\sigma'_\vartheta(E)$ is given by eq. (2.36) (with substitution $\zeta \to \vartheta$).

2.24.5 Spectrum

In this case, $\sigma'_\vartheta(E)$ can be represented in the form

$$\sigma'_\vartheta(E) = \frac{2}{\kappa_0} \frac{B(E)}{(A(E)\cos\vartheta + 2\sin\vartheta)^2 + \pi^2 B^2(E)\cos^2\vartheta} \equiv \rho_\vartheta^2(E),$$
$$A(E) = \operatorname{Re}\omega_0(E),\ B(E) = \pi^{-1}\operatorname{Im}\omega_0(E) = (1/2)[1 + \tanh(\pi\tilde{e})] > 0.$$

We see that $\sigma'_\vartheta(E)$ is positive a.c. function on whole E-axis, such that we obtain: spectrum of $\hat{h}_{0\vartheta}$ is simple and continuous, $\operatorname{spec}\hat{h}_{0\vartheta} = \mathbb{R}$. The set of generalized eigenfunctions $\{U_{\vartheta E}(u) = \rho_\vartheta(E)U_\zeta(u;W),\ E \in \mathbb{R}\}$ forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

2.25 $\lambda = 0$

For $\lambda = 0$, eq. (2.26) is reduced to

$$(\partial_u^2 + u^{-2}/4 + W)\psi(u) = 0. \tag{2.38}$$

2.25.1 Useful solutions

We use the following solutions of eq. (2.38):

$$O_{1,00}(u;W) = (\kappa_0 u)^{1/2} J_0(Ku),$$
$$O_{2,00}(u;W) = \frac{\pi}{2}(\kappa_0 u)^{1/2}\left\{N_0(Ku) - \frac{2}{\pi}\left[\ln(K/2\kappa_0) - \psi(1)\right] J_0(Ku)\right\} =$$
$$= (\kappa_0 u)^{1/2} J_0(Ku)\ln(\kappa_0 u) + (\kappa_0 u)^{1/2} Q(z),\ z = Wu^2,$$
$$Q(z) = \sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{z}{4}\right)^k[\psi(1) - \psi(k+1)],\ Q(z) = O(z),\ z \to 0,$$
$$O_{3,00}(u;W) = -\frac{i\pi}{2}(\kappa_0 u)^{1/2} H_0^{(1)}(Ku) = -\omega_{00}(W) O_{1,00}(u;W) + O_{2,00}(u;W),$$
$$K = W^{1/2} = \sqrt{|W|}e^{i\varphi_W/2},\ \omega_{00}(W) = i\pi/2 + \psi(1) - \ln(K/2\kappa_0).$$

Note that $O_{1,00}(u;W)$ and $O_{2,00}(u;W)$ are real-entire in W.

Asymptotics, $u \to 0$ We have

$$O_{1,00}(u;W) = (\kappa_0 u)^{1/2}(1 + O(u^2)),\ O_{2,00}(u;W) = (\kappa_0 u)^{1/2}\ln(\kappa_0 u)(1 + O(u^2)),$$
$$O_{3,00}(u;W) = -\omega_{00}(W)(\kappa_0 u)^{1/2}(1 + O(u^2)) + (\kappa_0 u)^{1/2}\ln(\kappa_0 u)(1 + O(u^2)).$$

Asymptotics, $u \to \infty$, $\operatorname{Im} W > 0$ ($\operatorname{Im} K > 0$) or $W = 0$ We have

$$O_{1,00}(u;W) = \left(\frac{\kappa_0}{2\pi K}\right)^{1/2} e^{-i(Ku - \pi/4)}(1 + O(u^{-1})) = O(e^{u \operatorname{Im} K}),$$
$$O_{3,00}(u;W) = -i\left(\frac{\pi\kappa_0}{2K}\right)^{1/2} e^{i(Ku - \pi/4)}(1 + O(u^{-1})) = O(e^{-u \operatorname{Im} K}).$$

Wronskian We have

$$\mathrm{Wr}(O_{1,00}, O_{3,00}) = \kappa_0.$$

2.25.2 Symmetric operator $\hat{h}_{00}$

Symmetric operator $\hat{h}_{00}$ associated with s.a. differential operation $\check{h}_{00} = -\partial_u^2 - u^2/4$ is given by eq. (2.27).

2.25.3 Adjoint operator $\hat{h}_{00}^+ = \hat{h}_{00}^*$

Adjoint operator $\hat{h}_{00}^+ = \hat{h}_{00}^*$ is given by is given by eqs. (2.28) – (2.30) with substitution $\check{h}_{00}$, $\hat{h}_{00}$, and $\hat{h}_{00}^*$ for $\check{h}_0$, $\hat{h}_0$, and $\hat{h}_0^*$.

2.25.4 Self-adjoint hamiltonians $\hat{h}_{0\vartheta}$, guiding functional

There are a family of s.a. hamiltonians $\hat{h}_{00\theta}$ given by eqs. (2.32), (2.31) (with substitution $\zeta \to \theta$, $\hat{h}_{0\zeta} \to \hat{h}_{00\theta}$). A simple guiding functional $\Phi_\theta(\xi;W)$ is given by

$$\Phi_\theta(\xi;W) = \int_0^\infty U_\theta(u;W)\xi(u)du,\ \xi \in \mathbb{D}_\theta = D_r(\mathbb{R}_+) \cap D_{h_{00\theta}}.$$
$$.U_\theta(u;W) = O_{1,00}(u;W)\sin\theta + O_{2,00}(u;W)\cos\theta.$$

$U_\theta(u;W)$ is real-entire solution of eq.(2.38) satisfying the boundary condition (2.31).

The spectrum of $\hat{h}_{00\theta}$ is simple.

2.25.5 Green function $G_\theta(u,v;W)$, spectral function $\sigma_\theta(E)$

We find the Green function $G_\theta(u,v;W)$ as the kernel of the integral representation

$$\psi(u) = \int_0^\infty G_\theta(u,v;W)\eta(v)dv,\ \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{00\theta} - W)\psi(u) = \eta(u),\ \operatorname{Im} W > 0, \tag{2.39}$$

for $\psi \in D_{h_{00\theta}}$. General solution of eq. (2.39) (under condition $\psi \in L^2(\mathbb{R}_+)$) can be represented in the form

$$\psi(u) = aO_{3,00}(u;W) - \frac{1}{\kappa_0}O_{1,00}(u;W)\eta_3(W) + \frac{1}{\kappa_0}I(u),\ \eta_3(W) = \int_0^\infty O_{3,00}(v;W)\eta(v)dv,$$

$$I(u) = O_{1,00}(u;W)\int_0^u O_{3,00}(v;W)\eta(v)dv - O_{3,00}(u;W)\int_0^u O_{1,00}(v;W)\eta(v)dv,$$

$$I(u) = O\left(u^{3/2}\ln u\right),\ u \to 0.$$

A condition $\psi \in D_{h_{00\theta}}$, (i.e.$\psi$ satisfies the boundary condition (2.31)) gives

$$a = -\frac{\cos\theta}{\kappa_0\omega_\theta(W)}\eta_3(W),\ \omega_\theta(W) = \omega_{00}(W)\cos\theta + \sin\theta,$$

$$\begin{aligned}
&G_\theta(u,v;W) = \Omega_\theta(W)U_\theta(u;W)U_\theta(v;W)+\\
&+\frac{1}{\kappa_0}\begin{cases} \tilde{U}_\theta(u;W)U_\theta(v;W),\ u > v \\ U_\theta(u;W)\tilde{U}_\theta(v;W),\ u < v \end{cases},\\
&\Omega_\theta(W) \equiv \frac{\tilde{\omega}_\theta(W)}{\kappa_0\omega_\theta(W)},\ \tilde{\omega}_\theta(W) = \omega_{00}(W)\sin\theta - \cos\theta,\\
&\tilde{U}_\theta(u;W) = O_{1,00}(u;W)\cos\theta - O_{2,00}(u;W)\sin\theta,
\end{aligned} \tag{2.40}$$

where we used an equality

$$O_{3,00}(u;W) = -\tilde{\omega}_\theta(W)U_\theta(u;W) - \omega_\theta(W)\tilde{U}_\theta(u;W).$$

Note that the function $\tilde{U}_\theta(u;W)$ is real-entire in W and the last term in the r.h.s. of eq. (2.40) is real for $W = E$. For $\sigma'_\theta(E)$, we find

$$\sigma'_\theta(E) = \frac{1}{\pi}\operatorname{Im}\Omega_\theta(E + i0).$$

2.25.6 Spectrum

$E = p^2 \geq 0$**,** $K = p \geq 0$ In this case, we have $\omega_{00}(E) = i\pi/2 + \psi(1) - \ln(p/2\kappa_0)$, such that we find

$$\begin{aligned}
&\sigma'_\theta(E) = \frac{2}{\kappa_0}\frac{1}{[g(E)\cos\theta + 2\sin\theta]^2 + \pi^2\cos^2\theta} \equiv \rho^2_\theta(E),\\
&g(E) = 2\psi(1) - \ln(E/4\kappa_0^2),
\end{aligned}$$

the spectrum of $\hat{h}_{00\theta}$ is simple and continuous for all $E \geq 0$, $\operatorname{spec}\hat{h}_{00\theta} = \mathbb{R}_+$.

$E=-\tau^2<0$, $K=i\tau=e^{i\pi/2}\tau$ In this case, we have $\omega_{00}(E)=\psi(1)-\ln(\tau/2\kappa_0)$.

Let $\theta=\pm\pi/2$. In this case we find

$$\sigma_\theta'(E)=\frac{1}{\pi\kappa_0}\operatorname{Im}\omega_{00}(E+i0)=\frac{1}{\pi\kappa_0}\operatorname{Im}\omega_{00}(E)=0,$$

such that the spectrum points are absent on the negative E-semiaxis.

Let $|\theta|<\pi/2$. In this case, an expression for $\sigma_\theta'(E)$ can be represented in the form

$$\sigma_\theta'(E)=-\frac{1}{\pi\kappa_0\cos^2\theta}\operatorname{Im}\frac{1}{f_\theta(E+i0)}, \ .$$

Because $f_\theta(E)=\psi(1)-\ln(\tau/2\kappa_0)+\tan\theta$ is real, $\sigma_\theta'(E)$ can be different from zero only in the point $E_{(-)}(\theta)$,

$$f_\theta(E_{(-)}(\theta))=0,\ E_{(-)}(\theta)=-4\kappa_0^2e^{2(\psi(1)+\tan\theta)},$$

such that we have

$$\sigma_\theta'(E)=\frac{2|E_{(-)}(\theta)|}{\kappa_0\cos^2\theta}\delta(E-E_{(-)}(\theta)),\ .$$

Finally we obtain:

for $\theta=\pm\pi/2$, $\mathrm{spec}\hat{h}_{00\pm\pi/2}=\mathbb{R}_+$, the set $\{U_{\pm\pi/2E}(u)=\rho_{\pm\pi/2}(E)U_{\pi/2}(u;E)=\sqrt{u/2}J_0(pu)$, $E\in\mathbb{R}_+\}$ of generalized eigenfunctions of $\hat{h}_{00\pm\pi/2}$ forms the complete orthohonalized system in $L^2(\mathbb{R}_+)$;

for $\theta\in(-\pi/2,\pi/2)$, $\mathrm{spec}\hat{h}_{00\pm\pi/2}=\mathbb{R}_+\cup\{E_{(-)}(\theta)\}$, the set $\{U_{\theta E}(u)=\rho_\theta(E)U_\theta(u;E)$, $E\in\mathbb{R}_+,\ U_{\theta(-)}(u)=\sqrt{\frac{2|E_{(-)}(\theta)|}{\kappa_0\cos^2\theta}}U_\theta(u;E_{(-)}(\theta))\}$ of (generalized) eigenfunctions of $\hat{h}_{00\theta}$ forms the complete orthohonalized system in $L^2(\mathbb{R}_+)$.

Note that the results of this section can be obtained as limit $\lambda\to0$ of the corresponding results of previous subsec.3.

Indeed, we have

$$\lim_{\lambda\to0}O_{1,0}(u;W)|_{\lambda\neq0}=(\kappa_0u)^{1/2}\sum_{k=0}^\infty\frac{(-W/4\varkappa^2)^k}{k!}\frac{\varkappa^{2k}u^{2k}}{k!}=$$
$$=(\kappa_0u)^{1/2}\sum_{k=0}^\infty\frac{(-1)^k(Ku/2)^{2k}}{(k!)^2}=(\kappa_0u)^{1/2}J_0(Ku)=O_{1,00}(u;W).$$

$$\lim_{\lambda\to0}O_{2,0}(u;W)|_{\lambda\neq0}=\partial_\delta\left[(\kappa_0u)^{1/2+\delta}\sum_{k=0}^\infty\frac{\Gamma(\alpha_\delta+k)}{\Gamma(\alpha_\delta)}\frac{\Gamma(\beta_\delta}{\Gamma(\beta_\delta+k)}\frac{\varkappa^{2k}u^{2k}}{k!}\right]_{\lambda,\delta\to0}=$$
$$=(\kappa_0u)^{1/2}J_0(Ku)\ln(\kappa_0u)+(\kappa_0u)^{1/2}\sum_{k=1}^\infty\frac{(-1)^k(Ku/2)^{2k}}{(k!)^2}[\psi(1)-\psi(1+k)]=$$
$$=(\kappa_0u)^{1/2}[J_0(Ku)\ln(\kappa_0u)+Q(z)]=O_{2,00}(u;W)..$$

Further, we have for $\sigma_\vartheta'(E)$

i) $E>0$, $\ln(\kappa_0/\varkappa)=i\pi/4+\ln(\kappa_0|\lambda|^{-1/4})$, $\alpha=1/2-iE/4\sqrt{|\lambda|}=1/2+e^{-i\pi/2}E/4\sqrt{|\lambda|}\equiv 1/2+z$, $\psi(\alpha)=\ln\alpha-1/2\alpha+O(\alpha^{-2})=\ln z+O(z^{-2})=-i\pi/2+\ln(E/4)-(1/2)\ln|\lambda|+O(|\lambda|)$

$$\begin{aligned}&\omega_0(E)=i\pi-\ln(E/4\kappa_0^2)+2\psi(1)+O(|\lambda|)\implies\\&A(E)=2\psi(1)-\ln(E/4\kappa_0^2)+O(|\lambda|)=g(E)+O(|\lambda|),\\&B(E)=1+O(|\lambda|),\end{aligned}$$

$$\lim_{\lambda\to -0}\sigma'_\vartheta(E)|_{\lambda<0}=\frac{2}{\kappa_0}\frac{1}{(g(E)\cos\vartheta+2\sin\vartheta)^2+\pi^2\cos^2\vartheta}=$$
$$=\sigma'_\theta(E)|_{\theta=\vartheta}\,.$$

ii) $E=-\tau^2<0$, $\ln(\kappa_0/\varkappa)=i\pi/4+\ln(\kappa_0|\lambda|^{-1/4})$, $\alpha=1/2+i|E|/4\sqrt{|\lambda|}=1/2+e^{i\pi/2}E/4\sqrt{|\lambda|}\equiv 1/2+z$,

$\psi(\alpha)=\ln\alpha-1/2\alpha+O(\alpha^{-2})=\ln z+O(z^{-2})=i\pi/2+\ln(|E|/4)-(1/2)\ln|\lambda|+O(|\lambda|)$, $\operatorname{Im}\psi(\alpha)=\pi/2-2\varepsilon$, $\varepsilon=(\pi/2)\exp(-\pi|E|/2\sqrt{|\lambda|})$, $|\lambda|=\frac{\pi^2|E|^2}{2\ln^2(\pi/2\varepsilon)}$

$$\omega_0(E)=2\psi(1)-\ln(E/4\kappa_0^2)+O(|\lambda|)+2i\varepsilon,$$
$$A(E)=2\psi(1)-\ln(E/4\kappa_0^2)+O(|\lambda|),\ B(E)=2\varepsilon/\pi,$$

$$\sigma'_\vartheta(E)=\frac{1}{\pi\kappa_0}\frac{\varepsilon}{([\psi(1)-\ln(\tau/2\kappa_0)+O(|\lambda|]\cos\vartheta+\sin\vartheta)^2+\varepsilon^2\cos^2\vartheta}.$$

ii$_a$) $\vartheta=\pm\pi/2$

We have

$$\sigma'_{\vartheta=\pm\pi/2}(E)=\frac{\varepsilon}{\pi\kappa_0}\to 0=\sigma'_{\theta=\pm\pi/2}(E).$$

ii$_b$) $|\lambda|<\pi/2$

Represent $\sigma'_\vartheta(E)$ in the form

$$\sigma'_\vartheta(E)=\frac{1}{\pi\kappa_0\cos^2\vartheta}\frac{\varepsilon}{[f_\vartheta(E)+O(|\lambda|]^2+\varepsilon^2}.$$

If $E\neq E_{(-)}(\vartheta)$, then $\lim_{\lambda\to -0}\sigma'_\vartheta(E)|_{\lambda<0}=0$. Represent $\sigma'_\vartheta(E)$ in the form

$$\sigma'_\vartheta(E)=\frac{1}{\pi\kappa_0\cos^2\vartheta}\frac{\varepsilon}{[f'_\vartheta(E_{(-)}(\vartheta))\Delta+\Delta^2 b(\Delta)+O(|\lambda|)]^2+\varepsilon^2},$$

where $\Delta=E-E_{(-)}(\vartheta)$, $b(\Delta)=\Delta^{-2}[f_\vartheta(E)-f'_\vartheta(E_{(-)}(\vartheta))\Delta]$. It is easy to obtain

$$\lim_{\lambda\to -0}\sigma'_\vartheta(E)|_{\lambda<0}=\frac{1}{\kappa_0 f'_\vartheta(E_{(-)}(\vartheta))\cos^2\vartheta}\delta(\Delta)=.\frac{2|E_{(-)}(\vartheta)|}{\kappa_0\cos^2\vartheta}\delta(\Delta)=\sigma'_\theta(E)|_{\theta=\vartheta}\,.$$
$$\lim_{\lambda\to -0}\sigma'_m(E)|_{\lambda<0}=0.$$

iii) $E=0$

iii$_a$) $\vartheta=\pm\pi/2$

$$\sigma'_{\vartheta=\pm\pi/2}(+0)\big|_{\lambda=-0}=\sigma'_{\vartheta=\pm\pi/2}(E)\big|_{\lambda=-0}\Big|_{E\to +0}=\frac{1}{2\kappa_0}=\sigma'_{\theta=\pm\pi/2}(+0),$$
$$\sigma'_{\vartheta=\pm\pi/2}(0)\big|_{\lambda=-0}=\frac{1}{4\kappa_0},$$
$$\sigma'_{\vartheta=\pm\pi/2}(-0)\big|_{\lambda=-0}=\sigma'_{\vartheta=\pm\pi/2}(E)\big|_{\lambda=-0}\Big|_{E\to -0}=0=\sigma'_{\theta=\pm\pi/2}(-0).$$

iii$_b$) $|\vartheta|<\pi/2$

$$\sigma'_\vartheta(+0)|_{\lambda=-0}=\sigma'_\vartheta(E)|_{\lambda=-0}\big|_{E\to +0}=0=\sigma'_\theta(+0)|_{\theta=\vartheta}\,,$$
$$\sigma'_\vartheta(0)|_{\lambda=-0}=0,$$
$$\sigma'_\vartheta(-0)|_{\lambda=-0}=\sigma'_\vartheta(E)|_{\lambda=-0}\big|_{E\to -0}=0=\sigma'_\theta(-0)|_{\theta=\vartheta}\,.$$

3 Quantum two-dimensional Coulomb-like interaction on a plane

3.1 Preliminaries

Let we have
$$\Psi(x,\varphi)=x^{-1/2}\sum_{m\in\mathbb{Z}}e^{im\varphi_x/2}\psi_m(x),\ x\in\mathbb{R}_+,\ 0\le\varphi_x\le 4\pi.$$
The Schroedinger equation
$$(\check{H}-\mathcal{E})\Psi(x,\varphi)=0,\ \check{H}=-\partial_x^2-x^{-1}\partial_x-x^{-2}\partial_{\varphi_x}^2+gx^{-1},$$
is reduced to the radial equation
$$(\check{h}_m-\mathcal{E})\psi_m(x)=0,\ \check{h}_m=-\partial_x^2+(2x)^{-2}(m^2-1)+gx^{-1}, \tag{3.1}$$
or
$$\partial_x^2\psi_m(x)+(-\frac{m^2-1}{4x^2}-\frac{g}{x}+\mathcal{E})\psi_m(x)=0, \tag{3.2}$$
where $\hbar^2\mathcal{E}/2\mu$ is complex energy, $\hbar^2 g/2\mu$ is the Coulomb coupling constant, $\check{h}_m$ are radial differential operations,
$$\mathcal{E}=|\mathcal{E}|e^{i\varphi_{\mathcal{E}}},\ 0\le\varphi_{\mathcal{E}}\le\pi,\ \operatorname{Im}\mathcal{E}\ge 0.$$

It is convenient for our aims first to consider solutions more general equation
$$\partial_x^2\psi_m(x)+(-\frac{(m+\delta)^2-1}{4x^2}-\frac{g}{x}+\mathcal{E})\psi_m(x)=0,\ |\delta|<1. \tag{3.3}$$

Introduce a new variable
$$z=2Kx, K=\sqrt{-\mathcal{E}}=\sqrt{|\mathcal{E}|}e^{i(\varphi_{\mathcal{E}}-\pi)/2}=\sqrt{|\mathcal{E}|}\left[\sin(\varphi_{\mathcal{E}}/2)-i\cos(\varphi_{\mathcal{E}}/2)\right],$$
$\partial_x=2K\partial_z$, $\partial_x^2=4K^2\partial_z^2$, and new function $\phi(z)$, $\psi(x)=z^{1/2+|m+\delta|/2}e^{-z/2}\phi(z)$. Then we obtain

$$\begin{aligned}&z\partial_z^2\phi(z)+(\beta_\delta-z)\partial_z\phi(z)-\alpha_\delta\phi(z)=0.\\ &\alpha_\delta=1/2+|m+\delta|/2-w,\ w=-g/2K,\ \beta_\delta=1+|m+\delta|.\end{aligned} \tag{3.4}$$

Eq. (3.4) is the equation for confluent hypergeometric functions, in the terms of which we can express solutions of eq. (3.3). We will use the following solutions:

$$\begin{aligned}&C_{1,m,\delta}(x;\mathcal{E})=(\kappa_0x)^{1/2+|m+\delta|/2}e^{-z/2}\Phi(\alpha_\delta,\beta_\delta;z),\\ &C_{2,m,\delta}(x;\mathcal{E})=\Gamma^{-1}(\beta_{-,\delta})(\kappa_0x)^{1/2-|m+\delta|/2}e^{-z/2}\Phi(\alpha_{-,\delta},\beta_{-,\delta};z),\\ &C_{3,m,\delta}(x;\mathcal{E})=(\kappa_0x)^{1/2+|m+\delta|/2}e^{-z/2}\Psi(\alpha_\delta,\beta_\delta;z)=\\ &=\frac{\Gamma(-|m+\delta|)}{\Gamma(\alpha_{-,\delta})}C_{1,m,\delta}(x;\mathcal{E})+\frac{(\kappa_0/2K)^{|m+\delta|}\Gamma(|m+\delta|)\Gamma(\beta_{-,\delta})}{\Gamma(\alpha_\delta)}C_{2,m,\delta}(x;\mathcal{E}),\\ &\alpha_{-,\delta}=1/2-|m+\delta|/2-w,\ \beta_{-,\delta}=1-|m+\delta|,\end{aligned} \tag{3.5}$$

where κ_0 is parameter of dimensionality of inverse length introduced in the file <Osc2-...tex>. Using relation GradRy.9.212.1, we find

$$C_{1,m,\delta}(x;\mathcal{E})|_{K\to -K} = (\kappa_0 x)^{1/2+|m+\delta|/2}e^{z/2}\Phi(1/2+|m+\delta|/2+w,\beta_\delta;-z) = \\ = (\kappa_0 x)^{1/2+|m+\delta|/2}e^{-z/2}\Phi(\alpha_\delta,\beta_\delta;z) = C_{1,m,\delta}(x;\mathcal{E}).$$

Similarly, we obtain $C_{2,m,\delta}(x;\mathcal{E})|_{K\to -K} = C_{2,m,\delta}(x;\mathcal{E})$. We thus obtain that the functions $C_{1,m,\delta}(x;\mathcal{E})$ and $C_{2,m,\delta}(x;\mathcal{E})$ are real-entire in $\mathcal{E}$.

3.2 $|m|\geq 2$

3.3 Useful solutions

Represent $C_{3,m,\delta}$ in the form

$$C_{3,m,\delta}(x;\mathcal{E}) = B_{m,\delta}(\mathcal{E})C_{1,m,\delta}(x;\mathcal{E}) + \frac{(\kappa_0/2K)^{|m+\delta|}\Gamma(|m+\delta|)}{\Gamma(\alpha_\delta)}C_{4,m,\delta}(x;\mathcal{E}),$$
$$C_{4,m,\delta}(x;\mathcal{E}) = \Gamma(\beta_{-,\delta})\left[C_{2,m,\delta}(x;\mathcal{E}) - A_{m,\delta}(\mathcal{E})C_{1,m,\delta}(x;\mathcal{E})\right],$$
$$A_{m,\delta}(\mathcal{E}) = \frac{(\kappa_0/2K)^{-|m|}\Gamma(\alpha)}{\Gamma(\beta_\delta)\Gamma(\alpha_-)},\ \frac{\Gamma(\alpha)}{\Gamma(\alpha_-)} = (-1)^{|m|}(1-\alpha)_{|m|},\ (1+x)_{|m|} = (1+x)\cdots(|m|+x),$$
$$B_{m,\delta}(\mathcal{E}) = \frac{\Gamma(-|m+\delta|)}{\Gamma(\alpha_{-,\delta})} + \frac{(\kappa_0/2K)^{|m+\delta|}\Gamma(|m+\delta|)\Gamma(\beta_{-,\delta})}{\Gamma(\alpha_\delta)}A_{m,\delta}(\mathcal{E}) =$$
$$= \frac{\Gamma(-|m+\delta|)}{\Gamma(\alpha_-)}\left[\frac{\Gamma(\alpha_-)}{\Gamma(\alpha_{-,\delta})} - \frac{(\kappa_0/2K)^{\delta_m}\Gamma(\alpha)}{\Gamma(\alpha_\delta)}\right],$$
$$\alpha = \alpha_0 = 1/2+|m|/2-w,\ \alpha_- = \alpha_{-,0} = 1/2-|m|/2-w,\ \beta = \beta_0 = 1+|m|,$$

Consider $A_{m,\delta}(\mathcal{E})$ in more details.

Let $|m| = 2k-1$, $k\in\mathbb{N}$. We have

$$(\kappa_0/2K)^{-|m|} = (-4\mathcal{E}/\kappa_0^2)^k(\kappa_0/2K),\ (1-\alpha)_{|m|} = -\frac{g}{2K}(-4\mathcal{E})^{1-k}\prod_{l=1}^{k-1}(g^2+4\mathcal{E}l^2),$$
$$(\kappa_0/2K)^{-|m|}(1-\alpha)_{|m|} = -\kappa_0^{-|m|}g\prod_{l=1}^{k-1}(g^2+4\mathcal{E}l^2).$$

Let $|m| = 2k$, $k\in\mathbb{N}$. We have

$$(\kappa_0/2K)^{-|m|} = (-4\mathcal{E}/\kappa_0^2)^k,\ (1-\alpha)_{|m|} = (-4\mathcal{E})^{-k}\prod_{l=0}^{k-1}[g^2+4\mathcal{E}(l+1/2)^2],$$
$$(\kappa_0/2K)^{-|m|}(1-\alpha)_{|m|} = \kappa_0^{-|m|}\prod_{l=0}^{k-1}[g^2+4\mathcal{E}(l+1/2)^2],$$

such that we obtain that $A_{m,\delta}(\mathcal{E})$ is real-entire polynomial function of $\mathcal{E}$ and $C_{4,m,\delta}(x;\mathcal{E})$ is real-entire in $\mathcal{E}$.

We obtain the solutions of eq.(3.2) as the limit $\delta \to 0$ of the solutions of eq. (3.5):

$$
\begin{aligned}
& C_{1,m}(x;\mathcal{E}) = C_{1,m,0}(x;\mathcal{E}) = (\kappa_0 x)^{1/2+|m|/2} e^{-z/2} \Phi(\alpha,\beta;z), \\
& C_{4,m}(x;\mathcal{E}) = \lim_{\delta\to 0} C_{4,m,\delta}(x;\mathcal{E}), \\
& C_{3,m}(x;\mathcal{E}) = (\kappa_0 x)^{1/2+|m|/2} e^{-z/2} \Psi(\alpha,\beta;z) = \\
& = B_m(\mathcal{E}) C_{1,m}(x;\mathcal{E}) + C_m(\mathcal{E}) C_{4,m}(x;\mathcal{E}),\ C_m(\mathcal{E}) = \frac{(\kappa_0/2K)^{|m|}\Gamma(|m|)}{\Gamma(\alpha)}, \\
& B_m(\mathcal{E}) = B_{m,0}(\mathcal{E}) = \frac{(-1)^{|m|+1}}{2\Gamma(\beta)\Gamma(\alpha_-)} \left[\psi(\alpha_-) + \psi(\alpha) + 2\ln(2K/\kappa_0)\right].
\end{aligned}
$$

where we used relations.

$$
\begin{aligned}
& |m+\delta| = |m| + \delta_m,\ \delta_m = \delta \operatorname{sign} m,\ \Gamma(-|m+\delta|) = \frac{(-1)^{|m|+1}}{\delta_m \Gamma(\beta)}, \\
& \frac{\Gamma(\alpha_-)}{\Gamma(\alpha_{-,\delta})} = 1 + \delta_m \psi(\alpha_-)/2,\ \frac{\Gamma(\alpha)}{\Gamma(\alpha_\delta)} = 1 - \delta_m \psi(\alpha)/2,\ (\kappa_0/2K)^{\delta_m} = 1 + \delta_m \ln(\kappa_0/2K).
\end{aligned}
$$

3.3.1 Asymptotics, $x \to 0$

We have

$$
\begin{aligned}
& C_{1,m}(x;\mathcal{E}) = C_{1,\text{mas}}(x)(1 + O(x)), \\
& C_{4,m}(x;\mathcal{E}) = C_{4,\text{mas}}(x)\left(1 + O(x)\right), \\
& C_{3,m}(x;\mathcal{E}) = C_m(\mathcal{E}) C_{4,\text{mas}}(x)\left(1 + O(x)\right),\ \operatorname{Im}\mathcal{E} > 0, \\
& C_{1,\text{mas}}(x) = (\kappa_0 x)^{1/2+|m|/2},\ C_{4,\text{mas}}(x) = (\kappa_0 x)^{1/2-|m|/2}.
\end{aligned}
$$

3.3.2 Asymptotics, $x \to \infty$, $\operatorname{Im}\mathcal{E} > 0$ $(\operatorname{Re} K > 0)$

$$
\begin{aligned}
& C_{1,m}(x;\mathcal{E}) = \frac{\kappa_0^{1/2+|m|/2}(2K)^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)} x^{-w} e^{Kx}(1 + O(x^{-1})) = O(x^{-w} e^{x \operatorname{Re} K}), \\
& C_{3,m}(x;\mathcal{E}) = \kappa_0^{1/2+|m|/2}(2K)^{-\alpha} x^{w} e^{-Kx}(1 + O(x^{-1})) = O(x^{w} e^{-x \operatorname{Re} K})
\end{aligned}
$$

3.3.3 Wronskians

$$
\begin{aligned}
& \operatorname{Wr}(C_{1,m}, C_{4,m}) = -\kappa_0 |m|, \\
& \operatorname{Wr}(C_{1,m}, C_{3,m}) = -\kappa_0 |m| C_m(\mathcal{E}) = -\frac{\kappa_0 (\kappa_0/2K)^{|m|}\Gamma(\beta)}{\Gamma(\alpha)} = -\omega_m(\mathcal{E})
\end{aligned}
$$

3.4 Symmetric operator $\hat{h}_m$

For given a differential operation $\check{h}_m$ (3.1), we determine the following symmetric operator $\hat{h}_m$,

$$
\hat{h}_m : \left\{ \begin{array}{l} D_{h_m} = \mathcal{D}(\mathbb{R}_+), \\ \hat{h}_m \psi(x) = \check{h}_m \psi(x),\ \forall \psi \in D_{h_m} \end{array} \right. .
$$

3.5 Adjoint operator $\hat{h}_m^+ = \hat{h}_m^*$

$$\hat{h}_m^+ : \left\{ \begin{array}{l} D_{h_m^+} = D^*_{\check{h}_m}(\mathbb{R}_+) = \{\psi_*, \psi_*' \text{ are a.c. in } \mathbb{R}_+,\ \psi_*, \hat{h}_m^+\psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}_m^+\psi_*(x) = \check{h}_m\psi_*(x),\ \forall\psi_* \in D_{h_m^+} \end{array} \right. .$$

3.5.1 Asymptotics

I) $x \to \infty$

Because $V(x) \to 0$ for large x, we have: $[\psi_*, \chi_*](x) \to 0$ as $x \to \infty$, $\forall\psi_*, \chi_* \in D_{h_m^+}$.

II) $x \to 0$

Because $\check{h}_m\psi_* \in L^2(\mathbb{R})$, we have

$$\check{h}_m\psi_*(x) = [-\partial_x^2 + (2x)^{-2}(m^2 - 1) + gx^{-1}]\psi_*(x) = \eta(x),\ \eta \in L^2(\mathbb{R}_+).$$

or

$$(\check{h}_m - \mathcal{E}_0)\psi_*(x) = \tilde{\eta}(x),\ \tilde{\eta}(x) = \eta(x) - \mathcal{E}_0\psi_*(x),\ \tilde{\eta} \in L^2(\mathbb{R}_+),$$

where $\mathcal{E}_0$ is an arbitrary (but fixed) number with $\operatorname{Im}\mathcal{E}_0 > 0$. General solution of this equation can be represented in the form

$$\begin{aligned} \psi_*(x) &= a_1 C_{1,m}(x;\mathcal{E}_0) + a_2 C_{4,m}(x;\mathcal{E}_0) + I(x), \\ \psi_*'(x) &= a_1 C_{1,m}'(x;\mathcal{E}_0) + a_2 C_{4,m}'(x;\mathcal{E}_0) + I'(x), \end{aligned}$$

where

$$\begin{aligned} I(x) &= \frac{C_{4,m}(x;\mathcal{E}_0)}{\kappa_0|m|}\int_0^x C_{1,m}(y;\mathcal{E}_0)\tilde{\eta}(y)dy + \frac{C_{1,m}(x;\mathcal{E}_0)}{\kappa_0|m|}\int_x^{x_0} C_{4,m}(y;\mathcal{E}_0)\tilde{\eta}(y)dy, \\ I'(x) &= \frac{C_{4,m}'(x;\mathcal{E}_0)}{\kappa_0|m|}\int_0^x C_{1,m}(y;\mathcal{E}_0)\tilde{\eta}(y)dy + \frac{C_{1,m}'(x;\mathcal{E}_0)}{\kappa_0|m|}\int_x^{x_0} C_{4,m}(y;\mathcal{E}_0)\tilde{\eta}(y)dy. \end{aligned}$$

We obtain with the help of the CB-inequality: $I(x)$ is bounded at infinity and

$$I(x) = \left\{ \begin{array}{l} O(x^{3/2}),\ |m| \geq 3 \\ O(x^{3/2}\sqrt{\ln x}),\ |m| = 2 \end{array} \right. ,\ I'(x) = \left\{ \begin{array}{l} O(x^{1/2}),\ |m| \geq 3 \\ O(x^{1/2}\sqrt{\ln x}),\ |m| = 2 \end{array} \right. ,\ x \to 0.$$

The condition $\psi_*(x) \in L^2(\mathbb{R}_+)$ gives $a_1 = a_2 = 0$ for all $|m| \geq 2$ such that we find

$$\psi_*(x) = \left\{ \begin{array}{l} O(x^{3/2}),\ |m| \geq 3 \\ O(x^{3/2}\sqrt{\ln x}),\ |m| = 2 \end{array} \right. ,\ \psi_*'(x) = \left\{ \begin{array}{l} O(x^{1/2}),\ |m| \geq 3 \\ O(x^{1/2}\sqrt{\ln x}),\ |m| = 2 \end{array} \right. ,\ x \to 0, \tag{3.6}$$

and $\omega_{h_m^+}(\chi_*, \psi_*) = \Delta_{h_m^+}(\psi_*) = 0$.

3.6 Self-adjoint hamiltonian $\hat{h}_{m\mathfrak{e}}$

Because $\omega_{h_m^+}(\chi_*, \psi_*) = \Delta_{h_m^+}(\psi_*) = 0$ (and also because $C_{1,m}(x;\mathcal{E})$ and $C_{3,m}(x;\mathcal{E})$ and their linear combinations are not s.-integrable for $\operatorname{Im}\mathcal{E} \neq 0$), the deficiency indices of initial symmetric operator $\hat{h}_m$ are zero, which means that $\hat{h}_{m\mathfrak{e}} = \hat{h}_m^+$ is a unique s.a. extension of the initial symmetric operator $\hat{h}_m$:

$$\hat{h}_{m\mathfrak{e}} : \left\{ \begin{array}{l} D_{h_{m\mathfrak{e}}} = D^*_{\check{h}_m}(\mathbb{R}_+) \\ \hat{h}_{m\mathfrak{e}}\psi(x) = \check{h}_m\psi(x),\ \forall\psi \in D_{h_{m\mathfrak{e}}} \end{array} \right. . \tag{3.7}$$

3.7 The guiding functional $\Phi(\xi;\mathcal{E})$

As a guiding functional $\Phi(\xi;\mathcal{E})$ we choose

$$\Phi(\xi;\mathcal{E}) = \int_0^\infty C_{1,m}(x;\mathcal{E})\xi(x)dx,\ \xi \in \mathbb{D} = D_r(\mathbb{R}_+) \cap D_{h_{m\mathfrak{e}}}.$$

$$D_r(a,b) = \{\psi(x):\ \mathrm{supp}\psi \subseteq [a,\beta_\psi],\ \beta_\psi < b\}.$$

The guiding functional $\Phi(\xi;\mathcal{E})$ is simple and the spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple.

3.8 Green function $G_m(x,y;\mathcal{E})$, spectral function $\sigma_m(E)$

We find the Green function $G_m(x,y;\mathcal{E})$ as the kernel of the integral representation

$$\psi(x) = \int_0^\infty G_m(x,y;\mathcal{E})\eta(y)dy,\ \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{m\mathfrak{e}} - \mathcal{E})\psi(x) = \eta(x),\ \mathrm{Im}\,\mathcal{E} > 0, \tag{3.8}$$

for $\psi \in D_{h_{m\mathfrak{e}}}$. General solution of eq. (3.8) can be represented in the form

$$\psi(x) = a_1 C_{1,m}(x;\mathcal{E}) + a_3 C_{3,m}(x;\mathcal{E}) + I(x),$$
$$I(x) = \frac{C_{1,m}(x;\mathcal{E})}{\omega_m(\mathcal{E})}\int_x^\infty C_{3,m}(y;\mathcal{E})\eta(y)dy + \frac{C_{3,m}(x;\mathcal{E})}{\omega_m(\mathcal{E})}\int_0^x C_{1,m}(y;\mathcal{E})\eta(y)dy,$$
$$I(x) = \left\{\begin{array}{l} O\left(x^{3/2}\right),\ |m| \geq 3 \\ O\left(x^{3/2}\sqrt{\ln x}\right),\ |m| = 2 \end{array}\right.,\ x \to 0.$$

A condition $\psi \in L^2(\mathbb{R}_+)$ gives $a_1 = a_3 = 0$, such that we find

$$\begin{aligned} G_m(x,y;\mathcal{E}) &= \frac{1}{\omega_m(\mathcal{E})}\left\{\begin{array}{l} C_{3,m}(x;\mathcal{E})C_{1,m}(y;\mathcal{E}),\ x > y \\ C_{1,m}(x;\mathcal{E})C_{3,m}(y;\mathcal{E}).\ x < y \end{array}\right. = \\ &= \Omega_m(\mathcal{E})C_{1,m}(x;\mathcal{E})C_{1,m}(y;\mathcal{E}) + \frac{1}{\kappa_0|m|}\left\{\begin{array}{l} C_{4,m}(x;\mathcal{E})C_{1,m}(y;\mathcal{E}),\ x > y \\ C_{1,m}(x;\mathcal{E})C_{4,m}(y;\mathcal{E}),\ x < y \end{array}\right., \end{aligned} \tag{3.9}$$

$$\Omega_m(\mathcal{E}) \equiv \frac{B_m(\mathcal{E})}{\omega_m(\mathcal{E})} = D_m(\mathcal{E})[2\ln(\kappa_0/2K) - \psi(\alpha) - \psi(\alpha_-)],$$
$$D_m(\mathcal{E}) = \frac{(2K/\kappa_0)^{|m|}(1-\alpha)_{|m|}}{2\kappa_0\Gamma^2(\beta)} =$$
$$= \frac{1}{2\kappa_0^{|m|+1}\Gamma^2(\beta)} \times \left\{\begin{array}{l} -g\prod_{l=1}^{k-1}((g^2 + 4\mathcal{E}l^2),\ |m| = 2k-1, \\ \prod_{l=0}^{k-1}[g^2 + 4\mathcal{E}(l+1/2)^2],\ |m| = 2k \end{array}\right.,\ k \in \mathbb{N},$$
$$\mathrm{Im}\,D_m(E) = 0,\ \forall E \in \mathbb{R}.$$

Note that the last term in the r.h.s. of eq. (3.9) is real for $\mathcal{E} = E$. From the relation

$$C_{1,m}^2(u_0;E)\sigma_m'(E) = \frac{1}{\pi}\mathrm{Im}\,G_m(u_0 - 0, u_0 + 0; E + i0),$$

we find

$$\sigma_m'(E) = \frac{1}{\pi}\mathrm{Im}\,\Omega_m(E + i0). \tag{3.10}$$

3.9 Spectrum

3.9.1 $E = p^2 \geq 0,\ p \geq 0,\ K = -ip = e^{-i\pi/2}p$

We have

$$
\begin{aligned}
&\alpha = 1/2 + |m|/2 - i\tilde{w},\ \alpha_- = 1/2 - |m|/2 - i\tilde{w},\ \tilde{w} = -g/2p,\\
&\operatorname{Im}\Omega_m(E) = D_m(E)[\pi - \operatorname{Im}(\psi(\alpha) + \psi(\alpha_-))] =\\
&= \pi D_m(E) \times \left\{ \begin{array}{l} \coth(\pi\tilde{w}) + 1,\ |m| = 2n-1 \\ 1 + \tanh(\pi\tilde{w}),\ |m| = 2n \end{array} \right. ,\ n \in \mathbb{N}.
\end{aligned}
$$

We thus find

$$
\begin{aligned}
&\sigma_m'(E) = \frac{1}{2\kappa_0^{|m|+1}\Gamma^2(\beta)} \times \left\{ \begin{array}{l} g[\coth(\pi g/2p) - 1] \prod_{l=1}^{k-1}((g^2 + 4El^2),\ |m| = 2k-1 \\ {[1 - \tanh(\pi g/2p)]} \prod_{l=0}^{k-1}[g^2 + 4E(l+1/2)^2],\ |m| = 2k \end{array} \right. =\\
&= \kappa_0^{-1-|m|} \frac{|\Gamma(\alpha)|^2 (2p)^{|m|} e^{-\pi g/2p}}{2\pi\Gamma^2(\beta)} \equiv \rho_m^2(E),\ \operatorname{spec}\hat{h}_{m\mathfrak{e}} = \mathbb{R}_+.
\end{aligned}
$$

3.9.2 $E = -\tau^2 < 0,\ \tau > 0,\ K = \tau$

We have

$$
\begin{aligned}
&\alpha = 1/2 + |m|/2 + g/2\tau,\ \alpha_- = 1/2 - |m|/2 + g/2\tau,\\
&\operatorname{Im}\Omega_m(E + i0) = -2D_m(E)\operatorname{Im}(\psi(\alpha)|_{\mathcal{E}=E+i0}\,.
\end{aligned}
$$

The function $\psi(\alpha)$ is real for $\mathcal{E} = E$ where $|\psi(\alpha)| < \infty$. Therefore, $\sigma_m'(E)$ can be not equal to zero only in the points $\psi(\alpha) = \pm\infty$, i. e., in the points $\alpha = -n$, $n \in \mathbb{Z}_+$.

$g \geq 0$ In this case, we have $\alpha > 0$ and the equation $\alpha = -n$ has no solutions, i.e., $\sigma_m'(E) = 0$.

$g < 0$ In this case, the equation $\alpha = -n$ has solutions,

$$
\begin{aligned}
&\tau_n = \frac{|g|}{1 + |m| + 2n},\ E_n = -\frac{g^2}{(1 + |m| + 2n)^2},\ n \in \mathbb{Z}_+,\\
&\operatorname{Im}\psi(\alpha) = -\sum_{n\in\mathbb{Z}_+} \frac{4\pi\tau_n^3}{|g|}\delta(E - E_n).
\end{aligned}
$$

We thus find

$$
\begin{aligned}
&\sigma_m'(E) = \sum_{n\in\mathbb{Z}_+} Q_n^2 \delta(E - E_n),\ Q_n = \sqrt{8|g|^{-1}\tau_n^3 D_m(E_n)} =\\
&= (2\tau_n/\kappa_0)^{1/2+|m|/2} \sqrt{\frac{2\tau_n(1+n)_{|m|}}{1 + |m| + 2n}},\ \operatorname{spec}\hat{h}_{m\mathfrak{e}} = \{E_n,\ n \in \mathbb{Z}_+\}.
\end{aligned}
$$

Finally, we find:

i) $g \geq 0$. The spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple and continuous, $\operatorname{spec}\hat{h}_{m\mathfrak{e}} = \mathbb{R}_+$.The set of the generalized eigenfunctions $\{U_{mE} = \rho_m(E)C_{1,m}(x;E),\ E \geq 0\}$ forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

ii) $g < 0$. The spectrum of $\hat{h}_{m\mathfrak{e}}$ is simple and has additionally the discrete part, $\mathrm{spec}\hat{h}_{m\mathfrak{e}} = \mathbb{R}_+ \cup \{E_n < 0,\ n \in \mathbb{Z}_+\}$. The set of the generalized eigenfunctions $\{U_{mE}(x) = \rho_m(E)C_{1,m}(x;E),\ E \geq 0\}$ and the eigenfunctions $\{U_{mn}(x) = Q_n C_{1,m}(x;E_n),\ n \in \mathbb{Z}_+\}$forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

3.10 $m = 1$

3.11 Useful solutions

We obtain the solutions of eq.(3.2) with $m = 1$ as the limit $\delta \to 0$ of the solutions of eq. (3.5) with $m = 1$:

$$
\begin{aligned}
&C_{1,1}(x;\mathcal{E}) = C_{1,1,0}(x;\mathcal{E}) = \kappa_0 x e^{-z/2}\Phi(\alpha, 2; z),\\
&C_{4,1}(x;\mathcal{E}) = \lim_{\delta\to 0} C_{4,1,\delta}(x;\mathcal{E}),\\
&C_{3,1}(x;\mathcal{E}) = \kappa_0 x e^{-z/2}\Psi(\alpha, 2; z) =\\
&\ = B_1(\mathcal{E})C_{1,1}(x;\mathcal{E}) + C_1(\mathcal{E})C_{4,1}(x;\mathcal{E}),\ C_1(\mathcal{E}) = \frac{\kappa_0}{2K\Gamma(\alpha)},\\
&B_1(\mathcal{E}) = B_{1,0}(\mathcal{E}) = \frac{1}{2\Gamma(\alpha_-)}\left[\psi(\alpha_-) + \psi(\alpha) + 2\ln(2K/\kappa_0)\right],\\
&\alpha = 1 - w = 1 + \frac{g}{2K},\ \alpha_- = -w = \frac{g}{2K}.
\end{aligned}
$$

3.11.1 Asymptotics, $x \to 0$

We have

$$
\begin{aligned}
&C_{1,1}(x;\mathcal{E}) = C_{1,1\mathrm{as}}(x)(1 + O(x)),\\
&C_{4,1}(x;\mathcal{E}) = C_{4,1\mathrm{as}}(x)\left(1 + O(x^2\ln x)\right),\\
&C_{3,m}(x;\mathcal{E}) = \left[B_1(\mathcal{E})C_{1,1\mathrm{as}}(x) + C_1(\mathcal{E})C_{4,1\mathrm{as}}(x)\right]\left(1 + O(x^2\ln x)\right),\ \operatorname{Im}\mathcal{E} > 0,\\
&C_{1,1\mathrm{as}}(x) = \kappa_0 x,\ C_{4,1\mathrm{as}}(x) = 1 + (\mathbf{C} - 1)gx + gx\ln(\kappa_0 x).
\end{aligned}
$$

It follows from these results that any solutions are s.-integrable at the origion.

3.11.2 Asymptotics, $x \to \infty$, $\operatorname{Im}\mathcal{E} > 0$ ($\operatorname{Re}K > 0$)

$$
\begin{aligned}
&C_{1,1}(x;\mathcal{E}) = \frac{\kappa_0(2K)^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)}x^{-w}e^{Kx}(1 + O(x^{-1})) = O(x^{-w}e^{x\operatorname{Re}K}),\\
&C_{3,1}(x;\mathcal{E}) = \kappa_0(2K)^{-\alpha}x^w e^{-Kx}(1 + O(x^{-1})). = O(x^w e^{-x\operatorname{Re}K})
\end{aligned}
$$

We see that the function $C_{3,1}(x;\mathcal{E})$ is s.-integrable for $\operatorname{Im}\mathcal{E} > 0$.

3.11.3 Wronskians

$$
\begin{aligned}
&\mathrm{Wr}(C_{1,m}, C_{4,m}) = -\kappa_0,\\
&\mathrm{Wr}(C_{1,m}, C_{3,m}) = -\kappa_0 C_1(\mathcal{E}) = -\frac{\kappa_0^2\Gamma(\beta)}{2K\Gamma(\alpha)} = -\omega_1(\mathcal{E})
\end{aligned}
$$

3.12 Symmetric operator $\hat{h}_1$

For given a differential operation $\check{h}_1$ (3.1), we determine the following symmetric operator $\hat{h}_1$,

$$\hat{h}_1 : \begin{cases} D_{h_1} = \mathcal{D}(\mathbb{R}_+), \\ \hat{h}_1\psi(x) = \check{h}_1\psi(x), \ \forall\psi \in D_{h_1} \end{cases} .$$

3.13 Adjoint operator $\hat{h}_1^+ = \hat{h}_1^*$

$$\hat{h}_1^+ : \begin{cases} D_{h_1^+} = D^*_{\check{h}_1}(\mathbb{R}_+) = \{\psi_*, \psi'_* \text{ are a.c. in } \mathbb{R}_+, \ \psi_*, \hat{h}_1^+\psi_* \in L^2(\mathbb{R}_+)\} \\ \hat{h}_1^+\psi_*(x) = \check{h}_1\psi_*(x), \ \forall\psi_* \in D_{h_1^+} \end{cases} .$$

3.13.1 Asymptotics

I) $x \to \infty$

Because $V(x) \to 0$ for large x, we have: $[\psi_*, \chi_*](x) \to 0$ as $x \to \infty$, $\forall\psi_*, \chi_* \in D_{h_1^+}$.

II) $x \to 0$

We represent the functions $\psi_* \in D_{h_1^+}$ in the form

$$\psi_*(x) = c_1C_{1,1}(x;\mathcal{E}_0) + c_2C_{4,1}(x;\mathcal{E}_0) + I(x),$$
$$\psi'_*(x) = c_1C'_{1,1}(x;\mathcal{E}_0) + c_2C'_{4,1}(x;\mathcal{E}_0) + I'(x),$$

where

$$I(x) = \frac{C_{4,1}(x;\mathcal{E}_0)}{\kappa_0}\int_0^x C_{1,m}(y;\mathcal{E}_0)\tilde{\eta}(y)dy - \frac{C_{1,1}(x;\mathcal{E}_0)}{\kappa_0}\int_0^x C_{4,m}(y;\mathcal{E}_0)\tilde{\eta}(y)dy,$$
$$I'(x) = \frac{C'_{4,1}(x;\mathcal{E}_0)}{\kappa_0}\int_0^u C_{1,1}(y;\mathcal{E}_0)\tilde{\eta}(y)dy - \frac{C'_{1,}(x;\mathcal{E}_0)}{\kappa_0}\int_0^x C_{4,1}(y;\mathcal{E}_0)\tilde{\eta}(y)dy.$$

We obtain with the help of the CB-inequality that

$$I(x) = O(x^{3/2}), \ I'(x) = O(x^{1/2}), \ x \to 0.$$

such that we find

$$\psi_*(x) = c_1C_{1,1\text{as}}(x) + c_2C_{4,1\text{as}}(x) + O(x^{3/2}),$$
$$\psi'_*(x) = c_1C'_{1,1\text{as}}(x) + c_2C'_{4,1\text{as}}(x) + O(x^{1/2}), \ x \to 0,$$

and

$$\Delta_{h_1^+}(\psi_*) = \kappa_0(\overline{c_2}c_1 - \overline{c_1}c_2) = i\kappa_0(\overline{c_+}c_+ - \overline{c_-}c_-),$$
$$c_\pm = \frac{1}{\sqrt{2}}(c_1 \pm ic_2).$$

3.14 Self-adjoint hamiltonians $\hat{h}_{1,\zeta}$

The condition $\Delta_{h_1^+}(\psi) = 0$ gives

$$
\begin{aligned}
&c_- = e^{2i\theta}c_+,\ 0 \le \theta \le \pi,\ \theta = 0 \sim \theta = \pi,\ \Longrightarrow \\
&c_1 \cos\zeta = c_2 \sin\zeta,\ \zeta = \theta - \pi/2,\ |\zeta| \le \pi/2,\ \zeta = -\pi/2 \sim \zeta = \pi/2,
\end{aligned}
$$

or

$$
\begin{aligned}
&\psi(x) = C\psi_{\rm as}(x) + O(x^{3/2}),\ \psi'(x) = C\psi'_{\rm as}(x) + O(x^{1/2}), \\
&\psi_{\rm as}(x) = C_{1,1\rm as}(x)\sin\zeta + C_{4,1\rm as}(x)\cos\zeta.
\end{aligned} \tag{3.11}
$$

We thus have a family of s.a.hamiltonians $\hat{h}_{1,\zeta}$,

$$
\hat{h}_{1,\zeta}: \left\{ \begin{array}{l} D_{h_{1\zeta}} = \{\psi \in D_{h_1^+},\ \psi \text{ satisfy the boundary condition (3.11)} \\ \hat{h}_{1,\zeta}\psi = \check{h}_1\psi,\ \forall\psi \in D_{h_{1,\zeta}} \end{array} \right. .
$$

3.15 The guiding functional $\Phi_{1,\zeta}(\xi;\mathcal{E})$

As a guiding functional $\Phi_{1,\zeta}(\xi;\mathcal{E})$ we choose

$$
\begin{aligned}
&\Phi_{1,\zeta}(\xi;\mathcal{E}) = \int_0^\infty U_{1,\zeta}(x;\mathcal{E})\xi(x)dx,\ \xi \in \mathbb{D}_{1,\zeta} = D_r(\mathbb{R}_+) \cap D_{h_{1,\zeta}}. \\
&.U_{1,\zeta}(x;\mathcal{E}) = C_{1,1}(x;\mathcal{E})\sin\zeta + C_{4,1}(x;\mathcal{E})\cos\zeta,
\end{aligned}
$$

$U_{1,\zeta}(u;\mathcal{E})$ is real-entire solution of eq. (3.2) (with $m = 1$) satisfying the boundary condition (3.11).

The guiding functional $\Phi_{1,\zeta}(\xi;\mathcal{E}W)$ is simple and the spectrum of $\hat{h}_\zeta$ is simple.

3.16 Green function $G_{1,\zeta}(x,y;\mathcal{E})$, spectral function $\sigma_{1,\zeta}(E)$

We find the Green function $G_{1,\zeta}(x,y;\mathcal{E})$ as the kernel of the integral representation

$$
\psi(x) = \int_0^\infty G_{1,\zeta}(x,y;\mathcal{E})\eta(y)dy,\ \eta \in L^2(\mathbb{R}_+),
$$

of unique solution of an equation

$$
(\hat{h}_{1,\zeta} - \mathcal{E})\psi(x) = \eta(x),\ \mathrm{Im}\,\mathcal{E} > 0, \tag{3.12}
$$

for $\psi \in D_{h_{1,\zeta}}$. General solution of eq. (3.12) can be represented in the form

$$
\begin{aligned}
&\psi(x) = a_1 C_{1,1}(x;\mathcal{E}) + a_3 C_{3,1}(x;\mathcal{E}) + I(x), \\
&I(x) = \frac{C_{1,1}(x;\mathcal{E})}{\omega_1(\mathcal{E})}\int_x^\infty C_{3,1}(y;\mathcal{E})\eta(y)dy + \frac{C_{3,1}(x;\mathcal{E})}{\omega_1(\mathcal{E})}\int_0^x C_{1,1}(y;\mathcal{E})\eta(y)dy, \\
&I(x) = \frac{C_{1,1as}(x;\mathcal{E})}{\omega_1(\mathcal{E})} + O\left(x^{3/2}\right),\ x \to 0.
\end{aligned}
$$

A condition $\psi \in L^2(\mathbb{R}_+)$ gives $a_1 = 0$. The condition $\psi \in D_{h_{1\zeta}}$, i.e., ψ satisfies the boundary condition (3.11), gives

$$\begin{aligned}
a_3 &= -\frac{1}{\omega_1}\tilde{a}_3 \int_0^\infty C_{3,1}(y;\mathcal{E})\eta(y)dy, \\
\tilde{a}_3 &= \frac{\cos\zeta}{C_1\omega_{1,\zeta}},\ \omega_{1,\zeta} = \omega_{1,\zeta}(\mathcal{E}) = f_1\cos\zeta - \sin\zeta, \\
f_1 &= f_1(\mathcal{E}) = \frac{B_1(\mathcal{E})}{C_1(\mathcal{E})}. = \frac{g}{2\kappa_0}[\psi(\alpha) + \psi(\alpha_-) + 2\ln(2K/\kappa_0)]
\end{aligned}$$

Using the relations

$$\begin{aligned}
&\tilde{a}_3 C_{3,1} - C_{1,1} = \frac{1}{\omega_{1,\zeta}} C_{1,\zeta}, \\
&C_{3,1} = C_1[\tilde{\omega}_{1,\zeta}C_{1,\zeta} + \omega_{1,\zeta}\tilde{C}_{1,\zeta}], \\
&C_{1,\zeta} = C_{1,1}\sin\zeta + C_{4,1}\cos\zeta, \\
&\tilde{C}_{1,\zeta} = C_{1,1}\cos\zeta - C_{4,1}\sin\zeta, \\
&\tilde{\omega}_{1,\zeta} = \tilde{\omega}_{1,\zeta}(\mathcal{E}) = f_1(\mathcal{E})\sin\zeta + \cos\zeta,\ \frac{\omega_1(\mathcal{E})}{C_1(\mathcal{E})} = \kappa_0,
\end{aligned}$$

we find

$$\begin{aligned}
&G_{1,\zeta}(x,y;\mathcal{E}) = \frac{1}{\omega_1}\left\{\begin{array}{l} C_{3,1}(x;\mathcal{E})[C_{1,1}(y;\mathcal{E}) - \tilde{a}_3 C_{3,1}(y;\mathcal{E})],\ x > y \\ [C_{1,m}(x;\mathcal{E}) - \tilde{a}_3 C_{3,1}(x;\mathcal{E})]C_{3,m}(y;\mathcal{E}).\ x < y \end{array}\right. = \\
&= -\frac{1}{\kappa_0}\Omega_{1,\zeta}(\mathcal{E})C_{1,\zeta}(x;\mathcal{E})C_{1,\zeta}(y;\mathcal{E}) - \frac{1}{\kappa_0}\left\{\begin{array}{l} \tilde{C}_{1,\zeta}(x;\mathcal{E})C_{1,\zeta}(y;\mathcal{E}),\ x > y \\ C_{1,\zeta}(x;\mathcal{E})C_{1,\zeta}(y;\mathcal{E}),\ x < y \end{array}\right. , \qquad (3.13) \\
&\Omega_m(\mathcal{E}) \equiv \frac{\tilde{\omega}_{1,\zeta}(\mathcal{E})}{\omega_{1,\zeta}(\mathcal{E})}.
\end{aligned}$$

Note that the last term in the r.h.s. of eq. (3.13) is real for $\mathcal{E} = E$. From the relation

$$C_{1,1}^2(x_0;E)\sigma_m'(E) = \frac{1}{\pi}\operatorname{Im} G_{1,\zeta}(x_0 - 0, x_0 + 0; E + i0),$$

we find

$$\sigma_{1,\zeta}'(E) = -\frac{1}{\pi\kappa_0}\operatorname{Im}\Omega_{1,\zeta}(E + i0).$$

3.17 Spectrum

3.17.1 $\zeta = \pi/2$

First we consider the case $\zeta = \pi/2$.

In this case, we have $U_{1,\pi/2}(x;\mathcal{E}) = C_{1,1}(x;\mathcal{E})$ and

$$\sigma_{1,\pi/2}'(E) = \frac{1}{\pi\kappa_0}\operatorname{Im} f_1(E + i0) = \frac{g}{2\pi\kappa_0^2}\operatorname{Im}[\psi(\alpha) + \psi(\alpha_-) + 2\ln(2K/\kappa_0)],$$

$E = p^2 \geq 0$, $p \geq 0$, $K = -ip = e^{-i\pi/2}p$ We have

$$\alpha = 1 - i\tilde{w},\ \alpha_- = -i\tilde{w},\ \tilde{w} = -g/2p,$$
$$\sigma'_{1,\pi/2}(E) = \frac{g}{2\pi\kappa_0^2}[\mathrm{Im}(\psi(\alpha) + \psi(\alpha_-)) - \pi] =$$
$$= \frac{g}{2\kappa_0^2}(\coth(\pi g/2p) - 1) \equiv \rho^2_{1,\pi/2}(E)$$

$E = -\tau^2 < 0$, $\tau > 0$, $K = \tau$ We have

$$\alpha = 1 + g/2\tau,\ \alpha_- = g/2\tau,$$
$$\sigma'_{1,\pi/2}(E) = \frac{g}{\pi\kappa_0^2}\,\mathrm{Im}\,\psi(\alpha)_{\mathcal{E}=E+i0}.$$

The function $\psi(\alpha)$ is real for $\mathcal{E} = E$ where $|\psi(\alpha)| < \infty$. Therefore, $\sigma'_{1,\pi/2}(E)$ can be not equal to zero only in the points $\psi(\alpha) = \pm\infty$, i. e., in the points $\alpha = -n$, $n \in \mathbb{Z}_+$.

$g \geq 0$ In this case, we have $\alpha > 0$ and the equation $\alpha = -n$ has no solutions, i.e., $\sigma'_m(E) = 0$.

$g < 0$ In this case, the equation $\alpha = -n$ has solutions,

$$\tau_n = \frac{|g|}{2(1+n)},\ \mathcal{E}_{1,n} = -\frac{g^2}{4(1+n)^2},\ n \in \mathbb{Z}_+,$$
$$\mathrm{Im}\,\psi(\alpha) = -\sum_{n\in\mathbb{Z}_+}\frac{4\pi\tau_n^3}{|g|}\delta(E - \mathcal{E}_{1,n}).$$

We thus find

$$\sigma'_{1,\pi/2}(E) = \sum_{n\in\mathbb{Z}_+} Q^2_{1,\pi/2,n}\delta(E - \mathcal{E}_{1,n}),\ Q_{1,\pi/2,n} = \frac{2\tau_n^{3/2}}{\kappa_0},$$
$$\mathrm{spec}\hat{h}_{1,\pi/2} = \{\mathcal{E}_{1,n},\ n \in \mathbb{Z}_+\}.$$

Finally, we find:

i) $g \geq 0$. The spectrum of $\hat{h}_{1,\pi/2}$ is simple and continuous, $\mathrm{spec}\hat{h}_{1,\pi/2} = \mathbb{R}_+$.The set of the generalized eigenfunctions $\{U_{1,\pi/2,E}(x) = \rho^2_{1,\pi/2}(E)C_{1,1}(x;E),\ E \geq 0\}$ forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

ii) $g < 0$. The spectrum of $\hat{h}_{1,\pi/2}$ is simple and has additionally the discrete part, $\mathrm{spec}\hat{h}_{1,\pi/2} = \mathbb{R}_+ \cup \{\mathcal{E}_{1,n} < 0,\ n \in \mathbb{Z}_+\}$. The set of the generalized eigenfunctions $\{U_{1,\pi/2,E}(x) = \rho^2_{1,\pi/2}(E)C_{1,1}(x;E),\ E \geq 0\}$ and the eigenfunctions $\{U_{1,\pi/2,n}(x) = Q_{1,\pi/2,n}C_{1,1}(x;\mathcal{E}_{1,n}),\ n \in \mathbb{Z}_+\}$forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

The same results for spectrum and eigenfunctions we obtain for the case $\zeta = -\pi/2$.

Note that all results for spectrum and eigenfunctions can be obtained from corresponding formulas of the sec 2 setting there $m = 1$.

3.17.2 $|\zeta| < \pi/2$

Now we consider the case $|\zeta| < \pi/2$.

In this case, we can represent $\sigma'_{1,\zeta}(E)$ in the form

$$\sigma'_{1,\zeta}(E) = -\frac{1}{\pi\kappa_0\cos^2\zeta}\operatorname{Im}\frac{1}{f_{1,\zeta}(E+i0)},\ f_{1,\zeta}(\mathcal{E}) = f_1(\mathcal{E}) - \tan\zeta.$$

$E = p^2 \geq 0$, $p \geq 0$, $K = e^{-i\pi/2}p$ In this case, we find

$$\begin{aligned}
&\sigma'_{1,\zeta}(E) = \frac{1}{\pi\kappa_0}\frac{B_1(E)}{(A_1(E)\cos\zeta - \sin\zeta)^2 + B_1^2(E)\cos^2\zeta}, \\
&A_1(E) = \operatorname{Re} f_1(E),\ B_1(E) = \operatorname{Im} f_1(E) = \frac{\pi g}{2\kappa_0}[\coth(\pi g/2p) - 1].
\end{aligned} \tag{3.14}$$

$g \leq 0$ In this case, $\sigma'_{1,\zeta}(E) > 0$ and is finite for $E > 0$, $\sigma'_{1,\zeta}(0) \geq 0$ and is finite for $|g| + |\zeta| \neq 0$, and $\sigma'_{1,\zeta}(E)$ has a behaviour $O(E^{-1/2})$ as $E \to 0$ for $g = \zeta = 0$, such that the spectrum of $\hat{h}_\zeta$ is simple and continuous, $\operatorname{spec}\hat{h}_{1,\zeta} = \mathbb{R}_+$.

$g > 0$ In this case, $\sigma'_{1,\zeta}(E)$ is given by eq. (3.14) for $E > 0$. But now, $B_1(0) = 0$ and the denominator of expr. (3.14) is equal to zero for $E = 0$ and $\zeta = \zeta_1$, $\tan\zeta_1 = A_1(0) = (g/\kappa_0)\ln(g/\kappa_0)$, such that we should study the behaviour of $\sigma'_{1,\zeta}(E)$ for small E in more details. We have

$$f_{1,\zeta}(\mathcal{E}) = (g/\kappa_0)\ln(g/\kappa_0) - \tan\zeta + \mathcal{E}/(3g^2) + O(\mathcal{E}^2).$$

We see that if $\zeta \neq \zeta_1$ then $\sigma'_{1,\zeta}(0) = 0$. But if $\zeta = \zeta_1$, we find

$$\sigma'_{1,\zeta}(E) = -\frac{3g^2}{\pi\kappa_0\cos^2\zeta_1}\operatorname{Im}\frac{1}{E+i0} + O(1) = \frac{3g^2}{\kappa_0\cos^2\zeta_1}\delta(E) + O(1),$$

and $\operatorname{spec}\hat{h}_{1,\zeta_1} = \mathbb{R}_+ \cup \{E_{(-)}(\zeta_1) = 0\}$.

$E = -\tau^2 < 0$, $\tau > 0$, $K = \tau$ In this case, we have

$$\begin{aligned}
f_1(E) &= \frac{g}{2\kappa_0}[\psi(1 + g/2\tau) + \psi(g/2\tau) + 2\ln(2\tau/\kappa_0)] = \\
&= \frac{g}{\kappa_0}[\psi(g/2\tau) + 2\ln(2\tau/\kappa_0) + \tau/g],
\end{aligned}$$

so that .$f_1(E)$ is real and $\sigma'_{1,\zeta}(E)$ can be not equal to zero only in the points $E_{1,n}(\zeta)$ which are solutions of an equation

$$f_{1,\zeta}(E_{1,n}(\zeta)) = 0 \text{ or } f_1(E_{1,n}(\zeta)) = \tan\zeta. \tag{3.15}$$

We thus obtain

$$\sigma'_{1,\zeta}(E) = \sum_{n\in\mathcal{N}_1} Q^2_{1,\zeta,n}\delta(E - E_{1,n}(\zeta)),\ Q_{1,\zeta,n} = \frac{1}{|\cos\zeta|}\sqrt{\frac{2}{\kappa_0 f_1'(E_{1,n}(\zeta))}},$$

where $\mathcal{N}_1$ is some subset of integers, $\mathcal{N}_1 \in \mathbb{Z}$, and $f'(E_{1,n}(\zeta)) > 0$. Because any $E < 0$ is solution of eq. (3.15) for some ζ, $f_1'(E) > 0$ for all E. Futhermore, we find:

$$\partial_\zeta E_{1,n}(\zeta) = [f_1'(E_{1,n}(\zeta))\cos^2\zeta]^{-1} > 0.$$

$g > 0$ In this case, we have: $f_1(E)$ is a smooth function on the interval $-\infty < E < 0$; $f_1(E)$ increases monotonically from $-\infty$ to $A(0) = (g/\kappa_0)\ln(g/\kappa_0)$ as E run from $-\infty$ to -0. We thus obtain: for $\zeta \in (\zeta_1, \pi/2)$, eq. (3.15) has no solutions; and for any fixed $\zeta \in (-\pi/2, \zeta_1)$, eq. (3.21) has one solution $E_{1,(-)}(\zeta) \in (-\infty, 0)$ monotonically increasing from $-\infty$ to -0 as ζ run from $-\pi/2 + 0$ t0 $\zeta_0 - 0$.

$g = 0$ In this case, we have $f_1(E) = -\tau/\kappa_0$ and eq. (3.15) has no solutions for $\zeta \in [0, \pi/2)$ and one solution $E_{1,(-)}(\zeta) = -\kappa_0^2 \tan^2\zeta$ for any $\zeta \in (-\pi/2, 0)$.

$g < 0$ In this case, we have: $f_1(E) = -\frac{|g|}{\kappa_0}[\psi(-|g|/2\tau) - \tau/|g| + \ln(2\tau/\kappa_0)]$; $f(\mathcal{E}_{1,n} \pm 0) = \mp\infty$, $n \in \mathbb{Z}_+$; in each interval $(\mathcal{E}_{0,n-1}, \mathcal{E}_{0,n})$, $n \in \mathbb{Z}_+$, $f_1(E)$ increases monotonically from $-\infty$ to ∞ as E run from $\mathcal{E}_{1,n-1} + 0$ to $\mathcal{E}_{1,n} - 0$; in each interval $(\mathcal{E}_{1,n-1}, \mathcal{E}_{1,n})$, $n \in \mathbb{Z}_+$, for any fixed $\zeta \in (-\pi/2, \pi/2)$, eq. (3.15) has one solution $E_{1,n}(\zeta)$ monotonically increasing from $\mathcal{E}_{1,n-1} + 0$ to $\mathcal{E}_{1,n} - 0$ as ζ run from $-\pi/2 + 0$ to $\pi/2 - 0$. Here we set $\mathcal{E}_{1,-1} = -\infty$. We find

$$
\mathcal{N}_1 = \begin{cases} \varnothing,\ g > 0,\ \zeta \in (\zeta_1, \pi/2),\ \text{or } g \geq 0,\ \zeta = \pm\pi/2,\ g = 0,\ \zeta \in [0, \pi/2) \\ (-),\ g > 0,\ \zeta \in (-\pi/2, \zeta_1] \\ (-),\ g = 0,\ \zeta \in (-\pi/2, 0) \\ \mathbb{Z}_+,\ g < 0,\ \zeta \in [-\pi/2, \pi/2] \end{cases},
$$

where, for completeness, the cases $\zeta = \pm\pi/2$ and $E = 0$ are included.

Note the relation

$$
\lim_{\zeta \to \pi/2} E_{1,n}(\zeta) = \lim_{\zeta \to -\pi/2} E_{1,n+1}(\zeta) = \mathcal{E}_{1,n},\ n \in \mathbb{Z}_+.
$$

Finally we obtain: the spectra of $\hat{h}_{1,\zeta}$ are simple, spec$\hat{h}_{1,\zeta} = \mathbb{R}_+ \cup \{E_{1,n}(\zeta) \leq 0,\ n \in \mathcal{N}_1\}$, the set $\{U_{1,\zeta,E}(x) = \rho_{1,\zeta}(E) U_{1,\zeta}(x; E),\ E \in \mathbb{R}_+;\ U_{1,\zeta,n}(x) = Q_{1,\zeta,n} U_{1,\zeta}(x; E_{1,n}(\zeta)\},\ n \in \mathcal{N}_1\}$ of (generalized) eigenfunctions of $\hat{h}_{1,\zeta}$ forms the complete orthohonalized system in $L^2(\mathbb{R}_+)$, where

$$
\rho_{1,\zeta}(E) = \begin{cases} \text{eq. (3.14) for } g \leq 0,\ E \geq 0; \\ \text{and } g > 0,\ E > 0; \\ \lim_{E \to +0} \rho_{1,\zeta_1}(E) \text{ for } g > 0,\ E = 0 \end{cases}.
$$

3.18 $m = -1$

Only modification which we must do is the following: the extension parameter for the case $m = -1$ should be considered as indendent of the extension parameter for the case $m = 1$. It is convenient to denote the extension parameter for the case $m = 1$ as $\zeta_{(1)}$ and for the case $m = -1$ as $\zeta_{(-1)}$.

3.19 $m = 0$

3.20 Useful solutions

For $m = 0$, eqs. (3.2) and3.3) are redused respectively to

$$
\partial_x^2 \psi_m(x) + (\frac{1}{4x^2} - \frac{g}{x} + \mathcal{E})\psi_m(x) = 0, \tag{3.16}
$$

and

$$\partial_x^2\psi_m(x)+(-\frac{\delta^2-1}{4x^2}-\frac{g}{x}+\mathcal{E})\psi_m(x)=0,\ |\delta|<1.$$

We will use the following solutions of eq. (3.16)

$$\begin{aligned}
&C_{1,0}(x;\mathcal{E})=C_{1,0,0}(x;\mathcal{E})=(\kappa_0 x)^{1/2}e^{-z/2}\Phi(\alpha,1;z),\\
&C_{2,0}(x;\mathcal{E})=\partial_\delta C_{1,0,\delta}(x;\mathcal{E})|_{\delta=+0}=\\
&\ =(\kappa_0 x)^{1/2}e^{-z/2}\ \partial_\delta\Phi(\alpha_\delta,\beta_\delta;z)|_{\delta=+0}+(1/2)C_{1,0}(x;\mathcal{E})\ln(\kappa_0 x),\\
&C_{3,0}(x;\mathcal{E})=C_{3,0,0}(x;\mathcal{E})=(\kappa_0 x)^{1/2}e^{-z/2}\Psi(\alpha,1;z)=\\
&\ =\frac{\omega_0(\mathcal{E})}{\Gamma(\alpha)}C_{1,0}(x;\mathcal{E})-\frac{2}{\Gamma(\alpha)}C_{2,0}(x;\mathcal{E}),\\
&\alpha=1/2-w,\ \omega_0(\mathcal{E})=2\psi(1)-\psi(\alpha)-\ln(2K/\kappa_0).
\end{aligned}$$

3.20.1 Asymptotics

$x\to 0$ We have

$$\begin{aligned}
&C_{1,0}(u;\mathcal{E})=(\kappa_0 x)^{1/2}(1+O(x)),\ C_{2,0}(x;\mathcal{E})=(1/2)(\kappa_0 x)^{1/2}\ln(\kappa_0 x)(1+O(x)),\\
&C_{3,0}(x;\mathcal{E})=\left[\frac{\omega_0(\mathcal{E})}{\Gamma(\alpha)}(\kappa_0 x)^{1/2}-\frac{1}{\Gamma(\alpha)}(\kappa_0 x)^{1/2}\ln(\kappa_0 x)\right](1+O(x)),\ \operatorname{Im}E>0.
\end{aligned}$$

$x\to\infty$, $\operatorname{Im}\mathcal{E}>0$ We have

$$\begin{aligned}
&C_{1,0}(x;\mathcal{E})=\frac{\kappa_0^{1/2}(2K)^{\alpha-1}}{\Gamma(\alpha)}x^{-w}e^{Kx}(1+O(x^{-1}))=O(x^{-w}e^{x\operatorname{Re}K}),\\
&C_{3,0}(x;\mathcal{E})=\kappa_0^{1/2}(2K)^{-\alpha}x^{w}e^{-Kx}(1+O(x^{-1})).=O(x^{w}e^{-x\operatorname{Re}K})
\end{aligned}$$

3.20.2 Wronskian

$$\operatorname{Wr}(C_{1,0},C_{2,0})=\kappa_0/2,\ \operatorname{Wr}(C_{1,0},C_{3,0})=-\frac{\kappa_0}{\Gamma(\alpha)}$$

3.21 Symmetric operator $\hat{h}_0$

For given a differential operation $\check{h}_0=-\partial_x^2-\frac{1}{4x^2}+\frac{g}{x}$ we determine the following symmetric operator $\hat{h}_0$,

$$\hat{h}_0:\left\{\begin{array}{l}D_{h_0}=\mathcal{D}(\mathbb{R}_+),\\ \hat{h}_0\psi(u)=\check{h}_0\psi(u),\ \forall\psi\in D_{h_0}\end{array}\right..$$

3.22 Adjoint operator $\hat{h}_0^+=\hat{h}_0^*$

$$\hat{h}_0^+:\left\{\begin{array}{l}D_{h_0^+}=D^*_{\check{h}_0}(\mathbb{R}_+)=\{\psi_*,\psi_*'\text{ are a.c. in }\mathbb{R}_+,\ \psi_*,\hat{h}_0^+\psi_*\in L^2(\mathbb{R}_+)\}\\ \hat{h}_0^+\psi_*(u)=\check{h}_0\psi_*(u),\ \forall\psi_*\in D_{h_0^+}\end{array}\right..$$

3.22.1 Asymptotics

I) $x \to \infty$

Because $V(x) \to 0$ as $x \to \infty$, we have: $\psi_*(x), \psi_*'(x), [\psi_*, \psi_*](x) \to 0$ as $x \to \infty$, $\forall \psi_* \in D_{h_0^+}$.

II) $x \to 0$

By the standard way, we obtain

$$
\begin{aligned}
&\psi_*(x) = c_1 u_{1\mathrm{as}}(x) + c_2 u_{2\mathrm{as}}(x) + O(x^{3/2} \ln x), \\
&\psi_*'(x) = c_1 u_{1\mathrm{as}}'(x) + c_2 u_{2\mathrm{as}}'(x) + O(x^{1/2} \ln x), \\
&u_{1\mathrm{as}}(x) = (\kappa_0 x)^{1/2},\ u_{2\mathrm{as}}(x) = (1/2)(\kappa_0 x)^{1/2} \ln(\kappa_0 x).
\end{aligned}
$$

For the asymmetry form $\Delta_{h_0^+}(\psi_*)$, we find

$$
\begin{aligned}
&\Delta_{h_0^+}(\psi_*) = (\kappa_0/2)(\overline{c_2} c_1 - \overline{c_1} c_2) = (i\kappa_0/2)(\overline{c_+} c_+ - \overline{c_-} c_-), \\
&c_\pm = \frac{1}{\sqrt{2}}(c_1 \pm i c_2).
\end{aligned}
$$

3.23 Self-adjoint hamiltonians $\hat{h}_{0,\zeta}$

The condition $\Delta_{h_0^+}(\psi) = 0$ gives

$$
\begin{aligned}
&c_- = e^{2i\theta} c_+,\ 0 \le \theta \le \pi,\ \theta = 0 \sim \theta = \pi,\ \Longrightarrow \\
&c_1 \cos\zeta = c_2 \sin\zeta,\ \zeta = \theta - \pi/2,\ |\zeta| \le \pi/2,\ \zeta = -\pi/2 \sim \zeta = \pi/2,
\end{aligned}
$$

or

$$
\begin{aligned}
&\psi(x) = C\psi_{\mathrm{as}}(x) + O(x^{3/2} \ln x),\ \psi'(x) = C\psi_{\mathrm{as}}'(x) + O(x^{1/2} \ln x), \\
&\psi_{\mathrm{as}}(x) = u_{1\mathrm{as}}(x) \sin\zeta + u_{2\mathrm{as}}(x) \cos\zeta.
\end{aligned}
\tag{3.17}
$$

We thus have a family of s.a.hamiltonians $\hat{h}_{0,\zeta}$,

$$
\hat{h}_{0,\zeta} : \begin{cases} D_{h_{0,\zeta}} = \{\psi \in D_{h_0^+},\ \psi \text{ satisfy the boundary condition (3.17)} \\ \hat{h}_{0,\zeta}\psi = \check{h}_0 \psi,\ \forall \psi \in D_{h_{0,\zeta}} \end{cases} .
$$

3.24 The guiding functional $\Phi_{0,\zeta}(\xi; \mathcal{E})$

As a guiding functional $\Phi_{0,\zeta}(\xi; \mathcal{E})$ we choose

$$
\begin{aligned}
&\Phi_{0,\zeta}(\xi; \mathcal{E}) = \int_0^\infty U_{0,\zeta}(x; \mathcal{E}) \xi(x) dx,\ \xi \in \mathbb{D}_{0,\zeta} = D_r(\mathbb{R}_+) \cap D_{h_{0,\zeta}}. \\
&.U_{0,\zeta}(x; \mathcal{E}) = C_{1,0}(x; \mathcal{E}) \sin\zeta + C_{2,0}(x; \mathcal{E}) \cos\zeta,
\end{aligned}
$$

$U_{0,\zeta}(u; \mathcal{E})$ is real-entire solution of eq. (3.2) (with $m = 0$) satisfying the boundary condition (3.17).

The guiding functional $\Phi_{0,\zeta}(\xi; \mathcal{E}W)$ is simple and the spectrum of $\hat{h}_{0,\zeta}$ is simple.

3.25 Green function $G_{0,\zeta}(x, y; \mathcal{E})$, spectral function $\sigma_{0,\zeta}(E)$

We find the Green function $G_{0,\zeta}(x, y; \mathcal{E})$ as the kernel of the integral representation

$$\psi(x) = \int_0^\infty G_{0,\zeta}(, y; \mathcal{E})\eta(y)dy, \ \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{0,\zeta} - \mathcal{E})\psi(x) = \eta(x), \ \operatorname{Im} \mathcal{E} > 0, \tag{3.18}$$

for $\psi \in D_{h_{0,\zeta}}$. General solution of eq. (3.18) (under condition $\psi \in L^2(\mathbb{R}_+)$) can be represented in the form

$$\begin{aligned}
&\psi(x) = aC_{3,0}(x; \mathcal{E}) + \frac{\Gamma(\alpha)}{\kappa_0} C_{1,0}(x; \mathcal{E})\eta_3(\mathcal{E}) + \frac{\Gamma(\alpha)}{\kappa_0} I(x), \ \eta_3(\mathcal{E}) = \int_0^\infty C_{3,0}(y; \mathcal{E})\eta(y)dy \\
&I(x) = C_{3,0}(x; \mathcal{E}) \int_0^x C_{1,0}(y; \mathcal{E})\eta(y)dy - C_{1,0}(x; \mathcal{E}) \int_0^x C_{3,0}(y; \mathcal{E})\eta(y)dy, \\
&I(x) = O\left(x^{3/2} \ln x\right), \ x \to 0.
\end{aligned}$$

A condition $\psi \in D_{h_{0,\zeta}}$(i.e.ψ satisfies the boundary condition (3.17)) gives

$$a = -\frac{\Gamma^2(\alpha)\cos\zeta}{2\kappa_0\omega_{0,\zeta}(\mathcal{E})}\eta_3(\mathcal{E}), \ \omega_{0,\zeta}(\mathcal{E}) = \frac{1}{2}\omega_0(\mathcal{E})\cos\zeta + \sin\zeta,$$

$$\begin{aligned}
&G_{0,\zeta}(x, y; \mathcal{E}) = \Omega_{0,\zeta}(\mathcal{E})U_{0,\zeta}(x; \mathcal{E})U_{0,\zeta}(y; \mathcal{E}) + \\
&+ \frac{2}{\kappa_0} \begin{cases} \tilde{U}_{0,\zeta}(x; \mathcal{E})U_{0,\zeta}(y; \mathcal{E}), \ x > y \\ U_{0,\zeta}(x; \mathcal{E})\tilde{U}_{0,\zeta}(y; \mathcal{E}), \ x < y \end{cases}, \\
&\Omega_{0,\zeta}(\mathcal{E}) \equiv \frac{2\tilde{\omega}_{0,\zeta}(\mathcal{E})}{\kappa_0\omega_{0,\zeta}(\mathcal{E})}, \ \tilde{\omega}_{0,\zeta}(\mathcal{E}) = \frac{1}{2}\omega_0(\mathcal{E})\sin\zeta - \cos\zeta, \\
&\tilde{U}_{0,\zeta}(x; \mathcal{E}) = C_{1,0}(x; \mathcal{E})\cos\zeta - C_{2,0}(x; \mathcal{E})\sin\zeta,
\end{aligned} \tag{3.19}$$

where we used an equality

$$\Gamma(\alpha)C_{3,0}(x; \mathcal{E}) = 2\tilde{\omega}_{0,\zeta}(\mathcal{E})U_{0,\zeta}(x; \mathcal{E}) + 2\omega_{0,\zeta}(\mathcal{E})\tilde{U}_{0,\zeta}(x; \mathcal{E}).$$

Note that the function $\tilde{U}_{0,\zeta}(x; \mathcal{E})$ is real-entire in $\mathcal{E}$ and the last term in the r.h.s. of eq. (3.19) is real for $W = E$. For $\sigma'_{0,\zeta}(E)$, we find

$$\sigma'_{0,\zeta}(E) = \frac{1}{\pi} \operatorname{Im} \Omega_{0,\zeta}(E + i0).$$

3.26 Spectrum

3.26.1 $\zeta = \pi/2$

First we consider the case $\zeta = \pi/2$.

In this case, we have $U_{0,\pi/2} = C_{1,0}(x; \mathcal{E})$ and

$$\sigma'_{0,\pi/2}(E) = -\frac{1}{\pi\kappa_0} \operatorname{Im}[\psi(\alpha) + \ln(2K/\kappa_0)],$$

$E = p^2 \geq 0,\ p \geq 0,\ K = e^{-i\pi/2}p$ In this case, we find

$$\sigma'_{0,\pi/2}(E) = \frac{1}{2\kappa_0} - \frac{1}{\pi\kappa_0}\operatorname{Im}[\psi(1/2 + ig/2p) = \frac{1}{2\kappa_0}[1 - \tanh(\pi g/2p)] \equiv$$
$$\equiv \rho^2_{0,\pi/2}(E),\ \operatorname{spec}\hat{h}_{0,\pi/2} = \mathbb{R}_+.$$

$E = -\tau^2 < 0,\ \tau > 0,\ K = \tau$ In this case, we have

$$\sigma'_{0,\pi/2}(E) = -\frac{1}{\pi\kappa_0}\operatorname{Im}\psi(\alpha),\ \alpha = 1/2 + g/2\tau.$$

$g \geq 0$ In this case, we have $\alpha > 0$, and $\sigma'_{0,\pi/2}(E) = 0$, spectrum points are absent.

$g < 0,\ \alpha = 1/2 - |g|/2\tau$ In this case, we have

$$\sigma'_{0,\pi/2}(E) = \sum_{n=0}^{\infty} Q^2_{0,\pi/2,n}\delta(E - \mathcal{E}_{0,n}),\ Q_{0,\pi/2,n} = \frac{2\tau_n}{\sqrt{\kappa_0(1+2n)}},$$
$$\tau_n = \frac{|g|}{1+2n},\ \mathcal{E}_{0,n} = -\frac{g^2}{(1+2n)^2}.$$

Finally:we obtain:

for $g \geq 0$, the spectrum $\hat{h}_{0,\pi/2}$ is simple and continuous, $\operatorname{spec}\hat{h}_{0,\pi/2} = \mathbb{R}_+$, and the set of generalized eigenfunctions $\{U_{0,\pi/2,E}(x) = \rho_{0,\pi/2}(E)C_{1,0}(x;E),\ E \in \mathbb{R}_+\}$ forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$;

for $g < 0$, the spectrum $\hat{h}_{0,\pi/2}$ is simple and contains continuous and discrete parts, $\operatorname{spec}\hat{h}_{0,\pi/2} = \mathbb{R}_+ \cup \{\mathcal{E}_{0,n} < 0,\ n \in \mathbb{Z}_+\}$, and the set of (generalized) eigenfunctions

$$\{U_{0,\pi/2,E}(x) = \rho_{0,\pi/2}(E)C_{1,0}(x;E),\ E \in \mathbb{R}_+; U_{0,\pi/2,n}(x) = Q_{0,\pi/2,n}C_{1,0}(x;\mathcal{E}_{0,n}),\ n \in \mathbb{Z}_+\}$$

forms a complete orthonormalized system in $L^2(\mathbb{R}_+)$.

Note that there results for spectrum and the set of eigenfunctions can be obtained from the corresponding results of sec. 2 by formal substitution $|m| \to 0$.

The same results we obtain for the case $\zeta = -\pi/2$.

3.26.2 $|\zeta| < \pi/2$

Now we consider the case $|\zeta| < \pi/2$.

In this case, we can represent $\sigma'_{0,\zeta}(E)$ in the form

$$\sigma'_{0,\zeta}(E) = -\frac{2}{\pi\kappa_0\cos^2\zeta}\operatorname{Im}\frac{1}{f_{0,\zeta}(E+i0)},\ f_{0,\zeta}(\mathcal{E}) = f_0(\mathcal{E}) + \tan\zeta,$$
$$f_0(\mathcal{E}) = \omega_0(\mathcal{E})/2.$$

$E = p^2 \geq 0,\ p \geq 0,\ K = e^{-i\pi/2}p$ In this case, we find

$$\sigma'_{0,\zeta}(E) = \frac{8}{\kappa_0}\frac{B_0(E)}{16(A_0(E)\cos\zeta + \sin\zeta)^2 + \pi^2 B_0^2(E)\cos^2\zeta}, \tag{3.20}$$
$$A_0(E) = \operatorname{Re} f_0(E),\ B_0(E) = (4/\pi)\operatorname{Im} f_0(E) = 1 - \tanh(\pi g/2p).$$

$g \leq 0$ In this case, $B_0(E) > 0$, $\forall E \geq 0$, and $\sigma'_{0,\zeta}(E) > 0$ and is finite, such that the spectrum of $\hat{h}_{0,\zeta}$ is simple and continuous, spec$\hat{h}_{0,\zeta} = \mathbb{R}_+$.

$g > 0$ In this case, $\sigma'_{0,\zeta}(E)$ is given by eq. (3.20) for $E > 0$. But now, $B_0(0) = 0$ and the denominator of expr. (3.20) is equal to zero for $E = 0$ and $\zeta = \zeta_0$, $\tan\zeta_0 = -A_0(0) = (1/2)\ln(g/\kappa_0) - \psi(1)$, such that we should study the behaviour of $\sigma'_{0,\zeta}(E)$ for small E in more details. We have

$$f_{0,\zeta}(\mathcal{E}) = \tan\zeta - [(1/2)\ln(g/\kappa_0) - \psi(1)] + \mathcal{E}/(12g^2) + O(\mathcal{E}^2).$$

We see that if $\zeta \neq \zeta_0$ then $\sigma'_{0,\zeta}(0) = 0$. But if $\zeta = \zeta_0$, we find

$$\sigma'_{0,\zeta}(E) = -\frac{24g^2}{\pi\kappa_0\cos^2\zeta_0}\operatorname{Im}\frac{1}{E + i0} + O(1) = \frac{24g^2}{\kappa_0\cos^2\zeta_0}\delta(E) + O(1),$$

and spec$\hat{h}_{0,\zeta_0} = \mathbb{R}_+ \cup \{E_{(-)}(\zeta_0) = 0\}$.

$E = -\tau^2 < 0$, $\tau > 0$, $K = \tau$ In this case, we have

$$f_0(E) = \psi(1) - \frac{1}{2}\psi(1/2 + g/2\tau) - \frac{1}{2}\ln(2\tau/\kappa_0),$$

so that .$f_0(E)$ is real and $\sigma'_{0,\zeta}(E)$ can be not equal to zero only in the points $E_{0,n}(\zeta)$ which are solutions of an equation

$$f_{0,\zeta}(E_{0,n}(\zeta)) = 0 \text{ or } f_0(E_{0,n}(\zeta)) = -\tan\zeta. \tag{3.21}$$

We thus obtain

$$\sigma'_{0,\zeta}(E) = \sum_{n\in\mathcal{N}_0} Q^2_{0,\zeta,n}\delta(E - E_{0,n}(\zeta)),\ Q_{0,\zeta,n} = \frac{1}{|\cos\zeta|}\sqrt{\frac{2}{\kappa_0 f'_0(E_{0,n}(\zeta))}},$$

where $\mathcal{N}_0$ is some subset of integers, $\mathcal{N}_0 \in \mathbb{Z}$, and $f'(E_{0,n}(\zeta)) > 0$. Because any $E < 0$ is solution of eq. (3.21) for some ζ, $f'_0(E) > 0$ for all E. Futhermore, we find:

$$\partial_\zeta E_{0,n}(\zeta) = -[f'_o(E_{0,n}(\zeta))\cos^2\zeta]^{-1} < 0.$$

$g > 0$ In this case, we have: $f_0(E)$ is a smooth function on the interval $-\infty < E < 0$; $f_0(E)$ increases monotonically from $-\infty$ to $A(0) = \psi(1) - \frac{1}{2}\ln(g/\kappa_0)$ as E run from $-\infty$ to -0. We thus obtain: for $\zeta \in (-\pi/2, \zeta_0)$, eq. (3.21) has no solutions; and for any fixed $\zeta \in (\zeta_0, \pi/2)$, eq. (3.21) has one solution $E_{0,(-)}(\zeta) \in (-\infty, 0)$ monotonically increasing from $-\infty$ to -0 as ζ run from $\pi/2 - 0$ t0 $\zeta_0 + 0$.

$g = 0$ In this case, we have $f_0(E) = \psi(1) - \frac{1}{2}\psi(1/2) - \frac{1}{2}\ln(2\tau/\kappa_0)$ and eq. (3.21) has one solution $E_{0,(-)}(\zeta) = (\kappa_0^2/4)e^{4\psi(1)-2\psi(1/2)+4\tan\zeta}$.

$g < 0$ In this case, we have: $f_0(E) = \psi(1) - \frac{1}{2}\psi(1/2 - |g|/2\tau) - \frac{1}{2}\ln(2\tau/\kappa_0)$; $f(\mathcal{E}_{0,n} \pm 0) = \mp\infty$, $n \in \mathbb{Z}_+$; in each interval $(\mathcal{E}_{0,n-1}, \mathcal{E}_{0,n})$, $n \in \mathbb{Z}_+$, $f_0(E)$ increases monotonically from $-\infty$ to ∞ as E run from $\mathcal{E}_{0,n-1} + 0$ to $\mathcal{E}_{0,n} - 0$; in each interval $(\mathcal{E}_{0,n-1}, \mathcal{E}_{0,n})$, $n \in \mathbb{Z}_+$, for any fixed $\zeta \in (-\pi/2, \pi/2)$, eq. (3.21) has one solution $E_{0,n}(\zeta)$ monotonically increasing from $\mathcal{E}_{0,n-1} + 0$ to $\mathcal{E}_{0,n} - 0$ as ζ run from $\pi/2 - 0$ t0 -$\pi/2 + 0$. Here we set $\mathcal{E}_{0,-1} = -\infty$. We find

$$\mathcal{N}_0 = \begin{cases} \varnothing,\ g > 0,\ \zeta \in (-\pi/2, \zeta_0) \text{ or } g \geq 0,\ \zeta = \pm\pi/2 \\ (-),\ g > 0,\ \zeta \in [\zeta_0, \pi/2) \\ (-),\ g = 0,\ \zeta \in (-\pi/2, \pi/2) \\ \mathbb{Z}_+,\ g < 0,\ \zeta \in [-\pi/2, \pi/2] \end{cases},$$

where, for completeness, the cases $\zeta = \pm\pi/2$ and $E = 0$ are included.

Note the relation

$$\lim_{\zeta \to -\pi/2} E_{0,n}(\zeta) = \lim_{\zeta \to \pi/2} E_{0,n+1}(\zeta) = \mathcal{E}_{0,n},\ n \in \mathbb{Z}_+.$$

Note also that all results for spectrum (and for eigenfunctions) in the case $g = 0$ can be obtained by the formal limit $g \to 0$ in the cases $g > 0$ or $g < 0$.

Finally we obtain: the spectra of $\hat{h}_{0,\zeta}$ are simple, spec$\hat{h}_{0,\zeta} = \mathbb{R}_+ \cup \{E_{0,n}(\zeta) \leq 0,\ n \in \mathcal{N}_0\}$, the set $\{U_{0,\zeta,E}(x) = \rho_{0,\zeta}(E) U_{0,\zeta}(x; E),\ E \in \mathbb{R}_+;\ U_{0,\zeta,n}(x) = Q_{0,\zeta,n} U_{0,\zeta}(x; E_{0,n}(\zeta)\},\ n \in \mathcal{N}_0\}$ of (generalized) eigenfunctions of $\hat{h}_{0,\zeta}$ forms the complete orthohonalized system in $L^2(\mathbb{R}_+)$, where

$$\rho_{0,\zeta}(E) = \begin{cases} \text{eq. (3.20) for } g \leq 0,\ E \geq 0 \\ \text{and } g > 0,\ E > 0, \\ \lim_{E \to +0} \rho_{0,\zeta}(E),\ \text{for } g > 0,\ E = 0 \end{cases}.$$

3.27 List of coincidences

Make the following identifications

$$\begin{aligned} &u = \sqrt{x/\kappa_0},\ W = -4\kappa_0 g,\ \lambda = -4\kappa_0^2 \mathcal{E}, \\ &x = \kappa_0 u^2,\ \mathcal{E} = -\lambda/4\kappa_0^2,\ g = -W/4\kappa_0,\ \Rightarrow \end{aligned}$$

$$\begin{aligned} &K = \sqrt{\lambda}/2\kappa_0 = \varkappa^2/2\kappa_0,\ \sqrt{K} = \varkappa/\sqrt{2\kappa_0},\ \kappa_0^2/\varkappa^2 = \kappa_0/2K, \\ &z = \rho,\ \sqrt{g} = \sqrt{-W}/2\sqrt{\kappa_0}, \\ &w_C = -\frac{g}{2K} = \frac{W}{4\sqrt{\lambda}} = w_O,\ \alpha_C = \alpha_O,\ \alpha_{C-} = \alpha_{O-},\ \beta_C = \beta_O, \end{aligned}$$

$$\begin{aligned} &E_C > 0,\ K = -i\sqrt{E_C},\ \lambda < 0;\ \sqrt{\lambda} = -i\sqrt{|\lambda|}, \\ &\tilde{w}_C = -iw_C = i\frac{g}{2K} = -\frac{g}{2\sqrt{E_C}} = \frac{E_O}{4\sqrt{|\lambda|}} = -i\frac{E_O}{4\sqrt{\lambda}} = \tilde{w}_O \end{aligned}$$

3.28 $|m| \geq 1$

We find

$$C_{k,m}(x;\mathcal{E}) = (k_0 u)^{1/2} O_{k,m}(u;W),\ k = 1,4,3.$$

$$\begin{aligned}
&C_{Om}(W) = \frac{(\kappa_0^2/\varkappa^2)^{|m|}\Gamma(|m|)}{\Gamma(\alpha_O)} = \frac{(\kappa_0/2K)^{|m|}\Gamma(|m|)}{\Gamma(\alpha_C)} = C_{Cm}(\mathcal{E}),\\
&\omega_{Om}(W) = 2\kappa_0|m|C_{Om}(W) = 2\kappa_0|m|C_{Cm}(\mathcal{E}) = 2\omega_{Cm}(\mathcal{E}),\\
&B_{Om}(W) = \frac{(-1)^{|m|+1}}{2\Gamma(\beta_O)\Gamma(\alpha_{O-})}\left[\psi(\alpha_{O-}) + \psi(\alpha_O) - 4\ln(\kappa_0/\varkappa)\right] =\\
&= \frac{(-1)^{|m+1|}}{2\Gamma(\beta_C)\Gamma(\alpha_{C-})}\left[\psi(\alpha_{C-}) + \psi(\alpha_C) + 2\ln(2K/\kappa_0)\right] = B_{Cm}(\mathcal{E}).
\end{aligned}$$

$$\Omega_{Cm}(\mathcal{E}) \equiv \frac{B_{Cm}(\mathcal{E})}{\omega_{Cm}(\mathcal{E})} = 2\frac{B_{Om}(W)}{\omega_{Om}(W)} = 2\Omega_{Om}(W).$$

3.29 $m = 0$

$$\begin{aligned}
&C_{k,0}(x;\mathcal{E}) = (k_0 u)^{1/2} O_{k,0}(u;W),\ k = 1,2,3,\\
&U_{C\zeta}(x;\mathcal{E}) = (k_0 u)^{1/2} U_{O\zeta}(u;W),\ \tilde{U}_{C\zeta}(x;\mathcal{E}) = (k_0 u)^{1/2}\tilde{U}_{O\zeta}(u;W).
\end{aligned}$$

$$\begin{aligned}
&\omega_{C0}(\mathcal{E}) = 2\psi(1) - \psi(\alpha_C) - \ln(2K/\kappa_0) =\\
&= 2\psi(1) - \psi(\alpha_O) + 2\ln(\kappa_0/\varkappa) = \omega_{O0}(W),\\
&\omega_{C\zeta}(\mathcal{E}) = \omega_{O\zeta}(W),\ \tilde{\omega}_{C\zeta}(\mathcal{E}) = \tilde{\omega}_{O\zeta}(W).
\end{aligned}$$

4 Conclusions

As we found, two dimensional oscillator and coulomb problems on pseudoshpere are described by the same equations in terms of the variables α and β This means that each point of the spectra of one of these theories corresponds a point of the spectra of the other theory, i.e. there is one-to-one correspondence between points of the the planes E_O, λ and E_C, g.

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References

[1] V. Ter-Antonyan, *Dyon-Oscillator Duality*, arXiv:quant-ph/0003106

[2] A.Nersessian, V.M.Ter-Antonyan, *Anyons, Monopole and Coulomb Problem*, Phys. Atom. Nucl. 61 (1998) 1756-1761

[3] Ye. Hakobyan, V. Ter-Antonyan, *Quantum oscillator as 1D anyon*, arXiv:quant-ph/0002069

[4] I.V. Tyutin, G.V. Grigoryan, R.P. Grigoiyan, *About dual one-dimcnsional oscillator and Coulomb-like theories*, arXiv:quant-ph/1011.5359.

[5] A.Nersessian (private comunication).

[6] M.A. Naimark, *Part II: Linear differential operators in Hilbert space*, Frederick Ungar Publishing Co., New York, 1968